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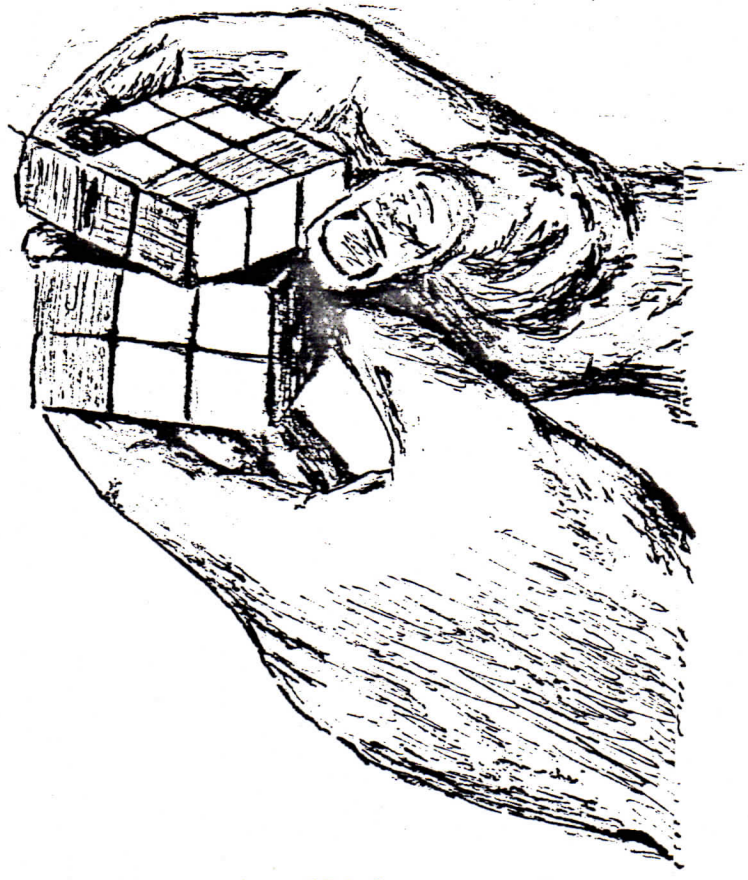
PUZZLESPASS MIT DEM RUBIK's CUBE



KERSTEN MEIER

This booklet is an improved translation of the German version "PUZZLESPASS MIT DEM RUBIK'S CUBE". I would like to thank Angelo Mastrocola, Paul Dennig and Margery Tate and the other members of the STANFORD RUBIK'S CUBE CLUB for helping to create this booklet which would not exist without their support.

January 2o, 1981

Kersten Meier

Preface

The Rubik's Cube is one of the most fascinating puzzles in the world. With its variaty of patterns there are many problems and solutions. You can deal with simple problems solving them in five minutes or tackle problems which take several weeks or month.

This booklet allows you to completely restore the cube from any starting configuration. It devides the solution in several handy parts. Each part of the solution is not revealed immediately. You are encouraged to work them out by yourself at first. It is hoped that this way you understand the solution better.

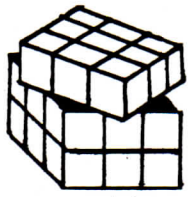
Every part of the problem is formulated exactly, supported by a drawing. In some case you might prefer not to solve a problem by yourself, perhaps after some trying it seems too difficult or too time consuming. Therefore the problem is divided into simpler ones which are easier to solve.

So every puzzler is offered problems of appropriate difficulty. After thorough study, every reader should be able to restore the cube. The summery of the solution system serves as a reference for all the sequences used troughout the text.

Introduction

Each of the six faces of the cube consists of nine little cubes called "cubies". In the restored cube each side consists of one color. By turning nine cubies in any layer, we can change the colors on a face.

Three types of moves are possible in each of the three axes of the cube : up, middle, down; front, middle, back; right, middle, left.



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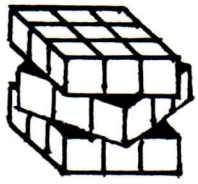

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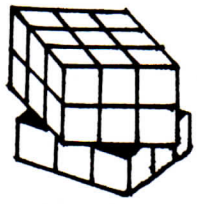
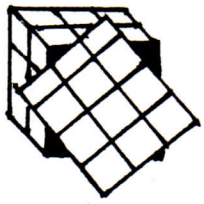
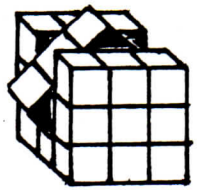


down



front



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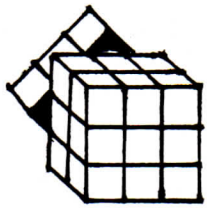
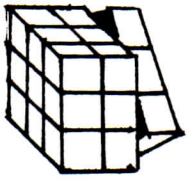
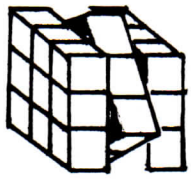


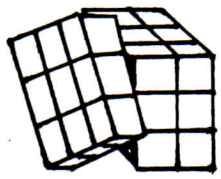
back



right



middle

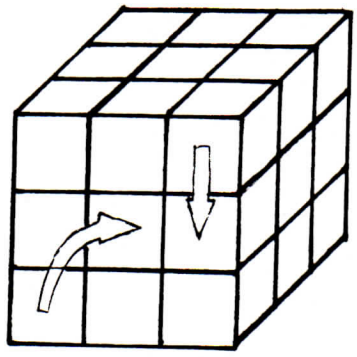


left

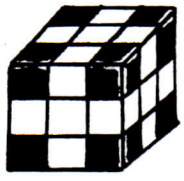
After a quarter, half or three-quarter move is done, one can choose another layer and turn it. This way the surfaces of the cube are shuffled.

The classical objective of the puzzle is to restore the cube so that each of the large faces is restored to its original color. At the first glance, one can think of turning each of the cubies into an arbitrary position. But by looking more closely, you will find that this is impossible ! A cubie from a corner of the cube can not be brought in the middle of a face or in the middle of an edge.

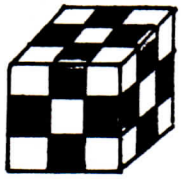
impossible



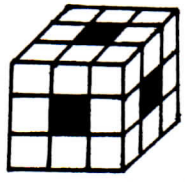
Among the cubies you can distinguish three different types : corner cubies, edge cubies and center cubies :



corner cubies



edge cubies



center cubies

There are eight corner cubies, twelve edge cubies and six center cubies. The corner cubies have three colored facelets, the edge cubies two, and the center cubies one. Because of this, it is impossible to bring a cubie of one type into the position of another.

Adding the number of cubies, you will find

$$8 + 12 + 6 = 26.$$

cubies. Actually the cube should consist of

$$3 \times 3 \times 3 = 27$$

cubies, but the cubie in the center of the whole cube is missing. It is replaced by a turning mechanism. In a later chapter we will take the cube apart and look at it.

Enough said about the cube - let's start puzzling !

I. Solution of the classical puzzle

We want to shuffle the cube and restore it to the original position. After puzzling for a couple of minutes you will notice that it is not easy to do.

1. Restoring the upper and middle layer

1.1 Restoring one face

We want to lower our aims a little bit and order only one face.

Problem 1 : Take the shuffled cube, choose any color, and bring all cubies of this color to one face of the large cube.

Solution to problem 1 :

You have probably chosen a face with several cubies of the same color on it. One can try to add cubies of the same color to this face.

Doing this you have to move already restored cubies to another face and bring them back later.

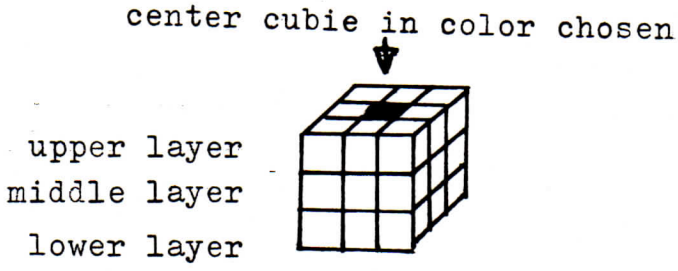
So the problem consists of keeping already ordered cubies in their positions or turning them away only for a while.

For problem 1 there are many different possibilities for a solution.

Therefore a simple and clear solution is described. We look at the center cubie of the color chosen and turn the other cubies of the same color on this face, step by step.

Hold the cube in a way that the center cubie chosen points up (Fig. 1). First, we want to position the four edge cubies,then the four corner cubies.

Fig. 1 :



First we deal with the upper layer. If there are already one or more edge cubies of the color chosen besides the center cubie, we can leave them where they are.

If one of the edge cubies sits next to the center cubie and the color chosen does not point up but rather to the side (Fig. 2) you have to flip the faces of the edge cubie (Fig. 3). After this we want to call faces of cubies facelets. Meanwhile , the dotted cubies are not allowed to be disturbed, for these cubies were possibly positoned earlier.

Fig. 2 :

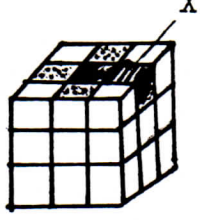
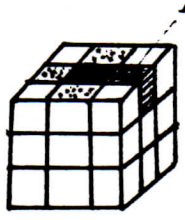


Fig. 3 :



Problem 1a : Flip the two facelets of the edge cubie X, without disturbing the dotted cubies.

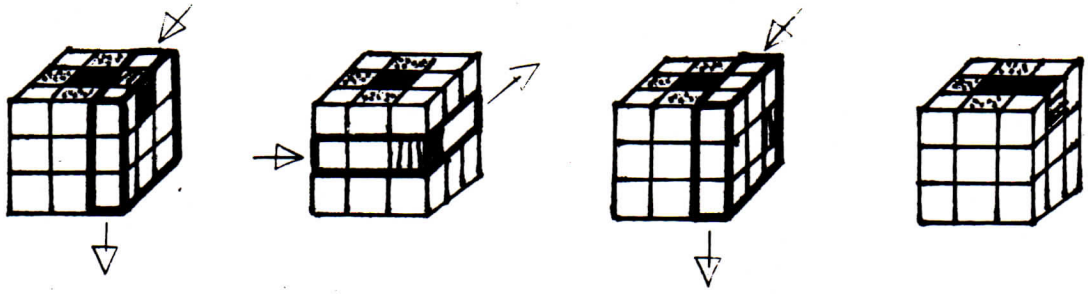
Preparation of the cube :

If the shuffled cube has no edge cubie in the position necessary for this problem, you should look at faces in the front, back, right or left. If there is any edge cubie pointing with one of the facelets in the color chosen to the side, you want to bring it up by moving this layer to the upper plane. We have the starting position for problem 1a and you can start puzzling.

Solution to problem 1a :

The problem can be solved with three moves (Fig. 4).

Fig. 4 :



Thick lines describe the layer in which next should be turned. The arrows show the direction of this turn. They are all quarter moves.

In this way, all approprate edge cubies being already in the upper layer can be restored.

Next, the edge cubies from the middle or lower layer in the color chosen shall be restored.

Problem 1b : Bring an edge cubie from the middle layer into position Y in Fig. 5 without disturbing the dotted cubies.

Problem 1c : Bring an edge cubie from the lower layer into position Y in Fig. 5 without disturbing the dotted cubies.

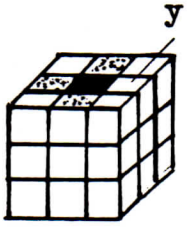


Fig. 5 :

Solution to problem 1b :

one can use the idea of the solution to problem 1a. Doing this you only have to move the edge cubie from the middle layer into the right layer : position a or b in Fig. 6.

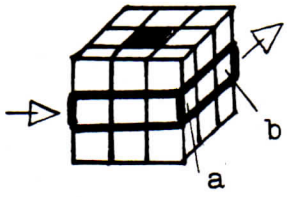


Fig. 6 :

Now the facelets with the color chosen should be pointing to the front or back, but not to the right. If it points to the right in one of both positions, then simply take the other position. The edge cubie only needs to be turned in the right layer to the top.

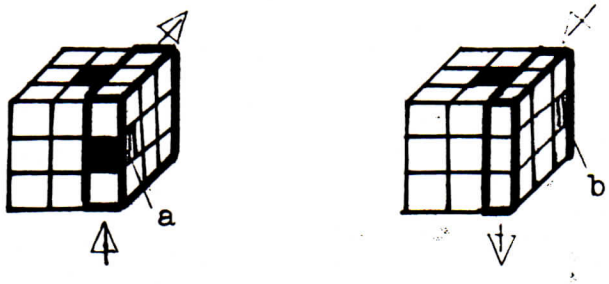


Fig. 7 :

Solution to problem 1c :

By turning the lower layer you can bring the edge cubie chosen into the right layer - position c in Fig. 8. By moving the right layer it goes into position Y.

Fig. 8 :

Eventually we have to flip the facelets of the edge cubie like in problem 1a.

Applying the solutions to problems 1a, 1b and 1c, all four edge cubies in the color chosen can be brought to the top.

In the next step, the four corner cubies of the top color shall be brought to the top, one after the other. One possible starting situation is shown in Fig. 9. The chosen color of the corner cubie in position a shows to the front.

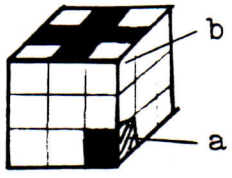


Fig. 9

Problem 1d : The corner cubie in position a shall be brought into position b with the chosen color to the top. The edge cubies in the upper layer which are already restored are allowed to be moved, but they must return into their old positions. Fig. 1o shows the desired position.

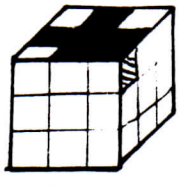
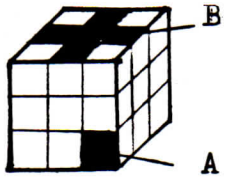


Fig. 1o

Fig. 11 :



Solution to problem 1d :

A possible solution is to bring the cubies A and B in Fig. 11 besides each other. To do this, the corner cubie A is turned out of the right layer, and the right layer is moved a quarter turn to the front. Now the first and second move are done in the opposite direction, and the desired position is obtained (Fig. 12)

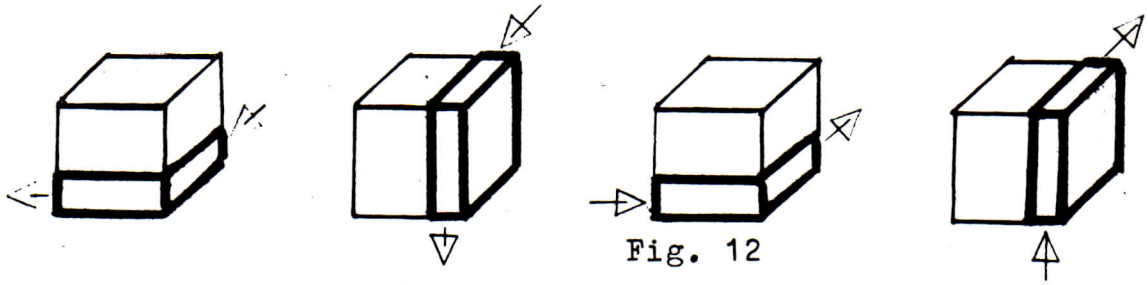
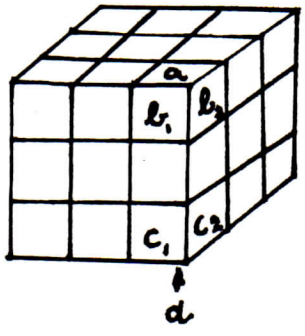


Fig. 12

Fig. 13 shows all positions where the corner cubie and its facelets with the colors chosen can be. A corner cubie that may have been in a different position in the lower layer was turned in the layer to the right front. The facelet of the corner cubie with the color chosen can be in position a , b_1, b_2, c_1, c_2 or d. In a, the corner cubie is already positioned correctly. Position c_1 was dealt with in problem 1d.

Fig. 13 :



Position c_2 is a reflection of position c_1. The solution to problem 1d has to be applied, in a reflected way. Fig. 14 shows the reflecting plane and its cut with the cube.

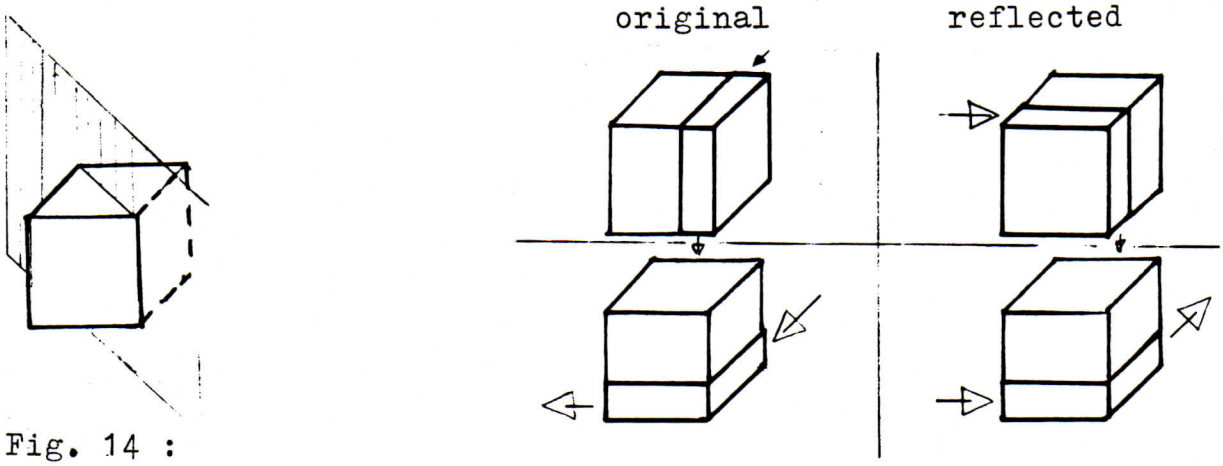


Fig. 14 :

Positions b_1 and b_2 are also reflected to each other and it is sufficient to deal with b_1.

Problem 1e : Twist the corner cubie A in Fig. 15, so that the color chosen points to the outside.

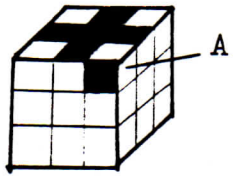


Fig. 15 :

Now only the case needs to be handled where the corner cubie is in the right lower front and the color chosen points down : position d in Fig. 13.

Problem 1f : Bring the corner cubie B whose facelet with the color chosen pointing downwards (Fig. 16) into position e with the color chosen to the top.

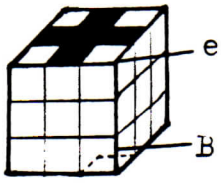


Fig. 16 :

Solution to problem 1e :

With four moves the starting position of problem 1d can be obtained :

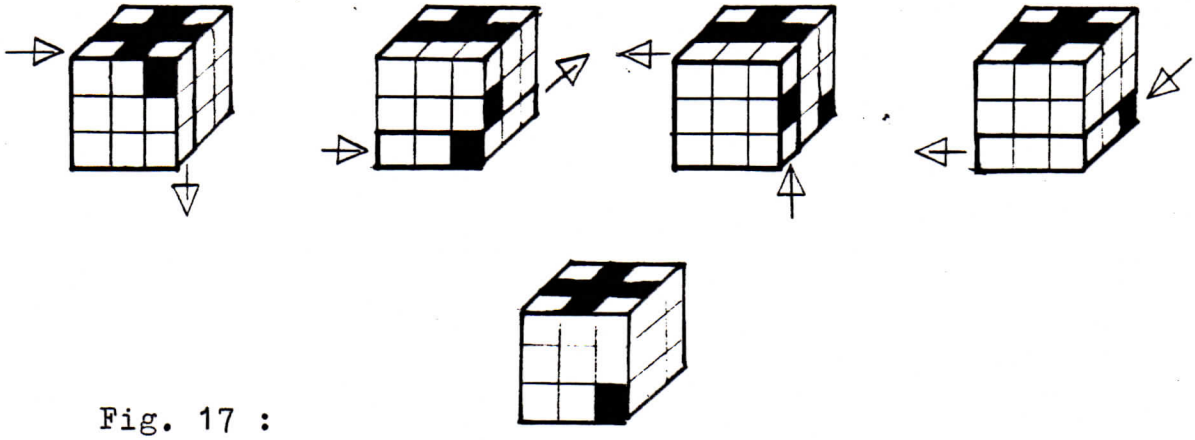


Fig. 17 :

The next four moves are executed like in the solution of 1d and we have gotten the desired pattern.

Solution to problem 1f :

Again the position of problem 1d can be reached with four moves.

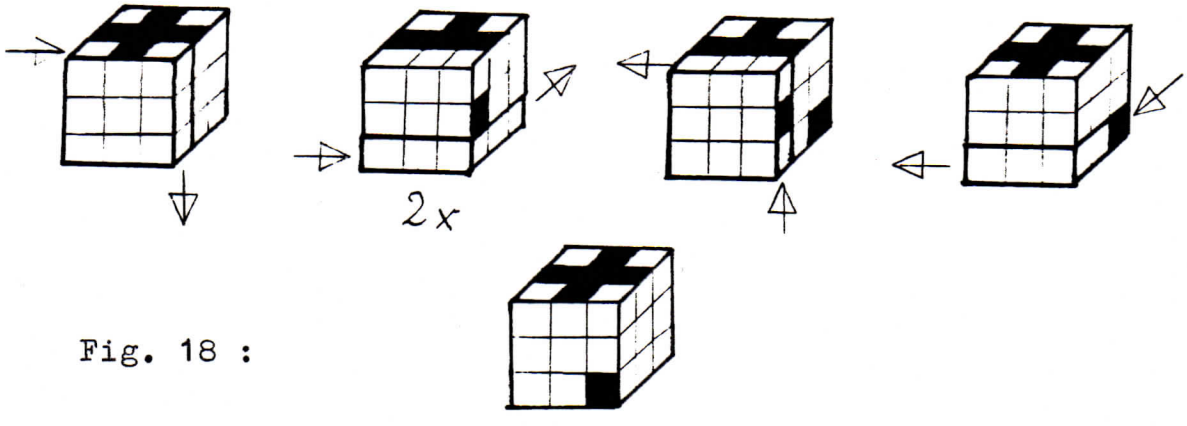


Fig. 18 :

2 x meansit is a 180^{o} turn. After this turn the same position is obtained like after two moves in the solution to problem 1e.

Having worked through the problems 1a to 1f there aren't any difficulties left to solve problem 1.

For practice you may want to rearrange a face several times, chosing a different color. After this it is easier to solve the problems in the next chapters.

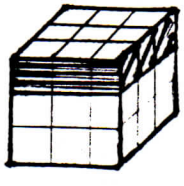
1.2 Restoring the upper layer

We know from the previous chapter how to restore one face. We might wish it would be possible to apply the moves used here for the other faces of the cube without disturbing the faces already restored. Then the classical problem would already be solved. Let's try it !

After only a few moves we see the sequences already discussed are not sufficient. Restoring the six faces one after the other - that is what we can do now. But every time we mix up the color previously arranged.

If we want to restore a face which is adjacent to the uniformly colored face, all three adjacent facelets have to get the same color. So in the upper layer three adjacent cubies of a face have to be colored uniformly as well (Fig. 19).

Fig. 19:



Problem 2 : Restore one face of the cube, so that all adjacent facelets of the same face have the same color (Fig.19).

Hint : You can solve problem 2 in two ways. In one way you use the surface already ordered like in the previous chapter. Alternatively, we can restore one face and its adjacent facelets simultaneously.

In chapter 1.1 we did not keep track of the adjacent facelets. Let us make up for that. Because we have already worked hard to restore one face, we prefer to keep it almost undisturbed by the following moves.

Solution to problem 2 :

Alternative A :

We have the cube with one face restored in front of us. Let us watch only the corner cubies in the upper layer. Are there already corner cubies with the same color on the same face (Fig. 2o) ?

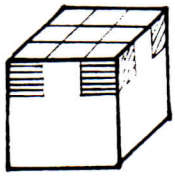
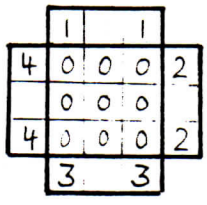


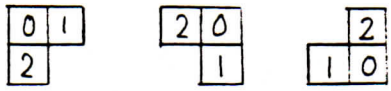
Fig. 2o

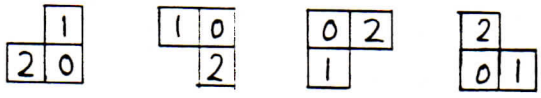
Fig. 21



If not, let us bring the four corner cubies into the correct position. Fig. 21 shows this desired position. Every color is symbolized by a number. 0 is a facelet on the face which is already restored. The corner cubies

can be positioned arbitrarely in the corners.

The cornes cubie  can be twisted :  ,

but you cannot exchange only two facelets. These patterns are impossible :  .

We want to direct the corner cubies into the position shown in Fig. 21. Actually there are only two different starting positions.

In the first case no two adjacent corner cubies match (Fig. 22).

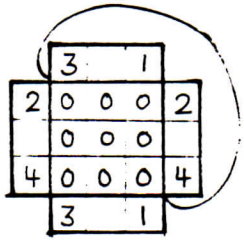


Fig. 22

Here two corner cubies have to be exchanged over the diagonal of the face already restored.

If there is a pair of adjacent corner cubies that are correctly positioned and a second pair that are not, then merely exchange the corner cubies of the second pair, for example the lower two in Fig. 23.

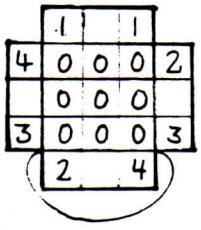


Fig. 23

If there are already three corner cubies correctly in place then the fourth must be correct, too.

Now we know what has to be exchanged in the two starting positions and we want to figure out to execute the exchanges.

Problem 2a : Exchange the two corner cubies in positions A and B, so that their original top facelets still point up. All other cubies in the upper layer - after temporary disturbance - return to their original positions.

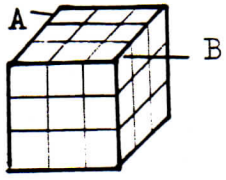


Fig. 24

Exchange along ~~an edge~~ a diagonal

Problem 2b : Exchange the two corner cubies in positions A and B in Fig. 25, so that their original top facelets still point up. All other cubies in the upper layer - after temporary disturbance - return to their original positions.

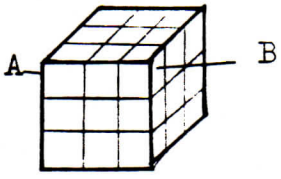
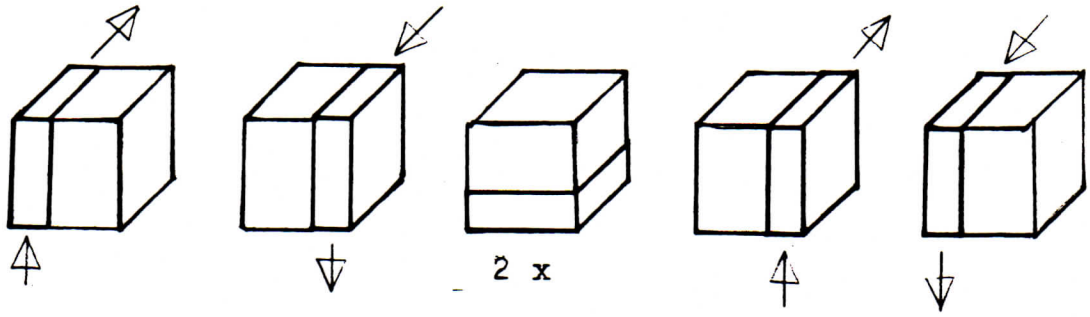


Fig. 25

Exchange along ~~a diagonal~~ an edge

Solution to problem 2a : (Exchange along ~~an edge~~ a diagonal)

There exists a very elegant and simple solution to this problem. The corner cubies to be exchanged are brought into the lower layer. Then this layer is turned by 180° and the positions of the observed corner cubies are exchanged. After retracing your first two moves backwards we have solved problem 2a.

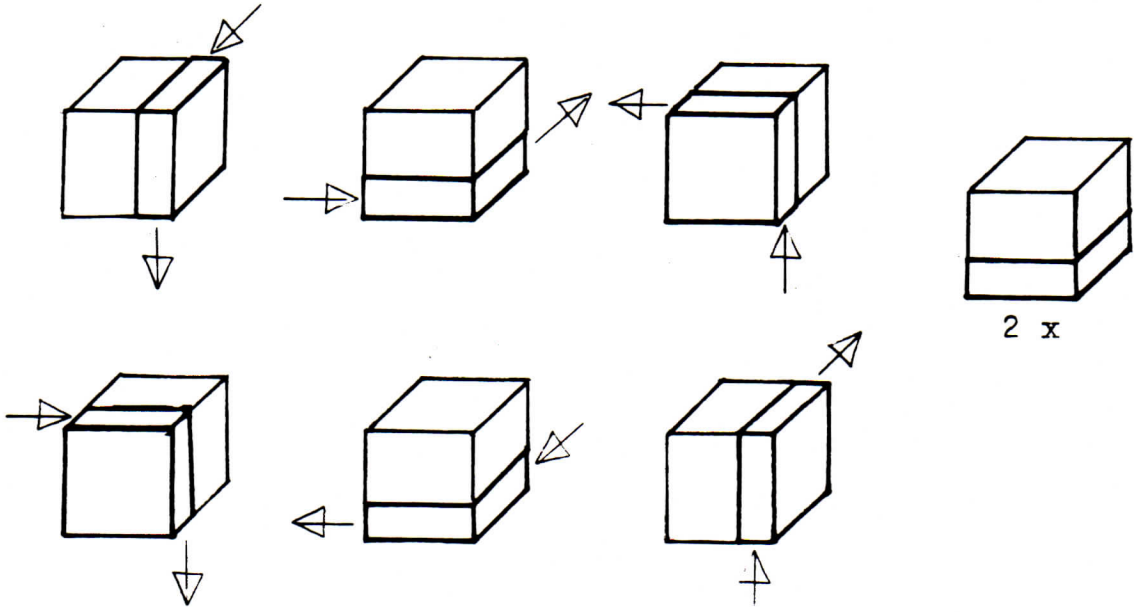


Let us label this sequence with five moves as symmetrical : The last two moves are executed in the reversed order and reversed direction of the first two. Such symmetries will appear again.

You can also solve the problem with sequences which are similar to those in chapter 1.1 to position the corner cubies. But you need more than five moves.

Solution to problem 2b : (Exchange along ~~a diagonal~~ an edge)

This problem has a symmetrical solution similar to that in problem 2a. It consists of seven moves. Again in the middle of the sequence there is a 180° turn of the bottom layer exchanging the two corner cubies.



Now the corner cubies in the upper layer hold their correct positions. Let us turn our attention to the edge cubies. Fig. 26 shows the correct positions.

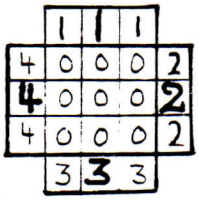


Fig. 26

Problem 2c : Given a uniform upper face (O) and correctly placed upper corner cubies, restore the edge cubies in the upper layer (Fig. 26).

Solution to problem 2c :

First of all let us deal with two special cases to problem 2c.

Problem 2d : Exchange the two edge cubies A and B in Fig. 27.

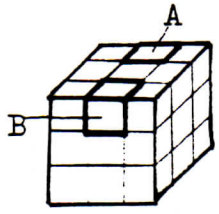


Fig. 27

Problem 2e : Exchange the two edge cubies A and B in Fig. 28.

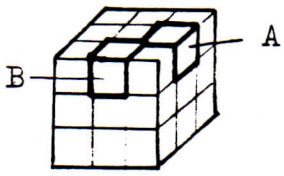
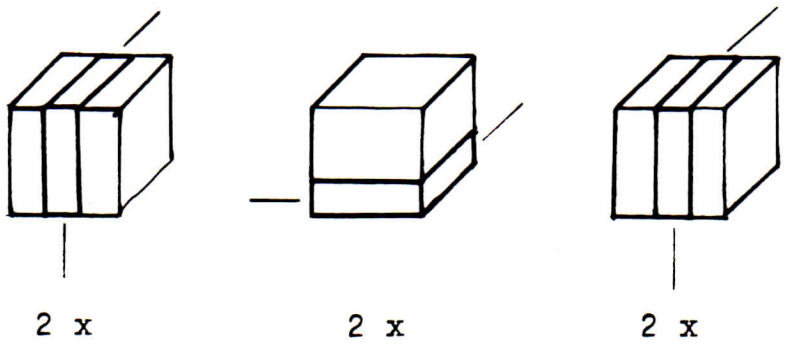


Fig. 28

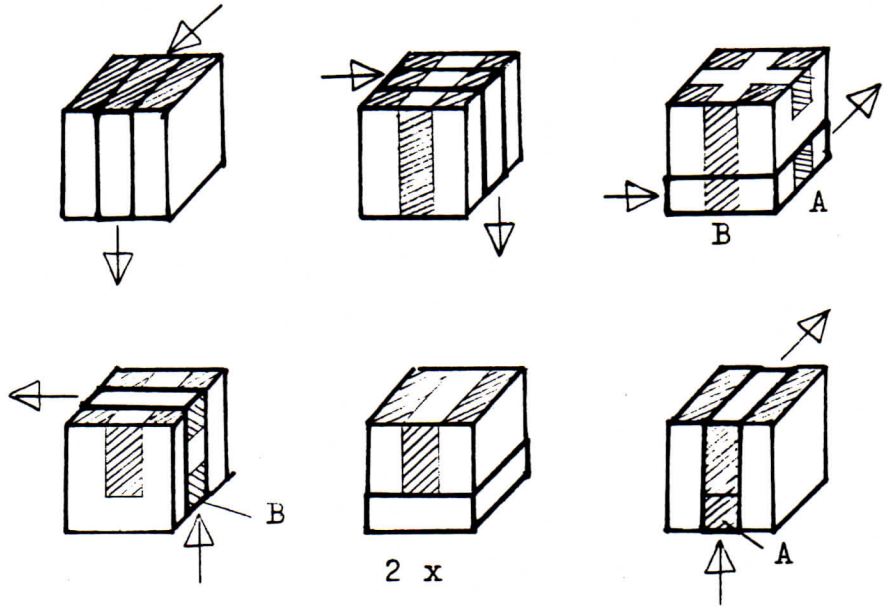
Solution to problem 2d :

For this problem we have a symmetrical solution with three moves. Note that, again, the middle of the sequence contains a 180° turn of the bottom layer.



Solution to problem 2d :

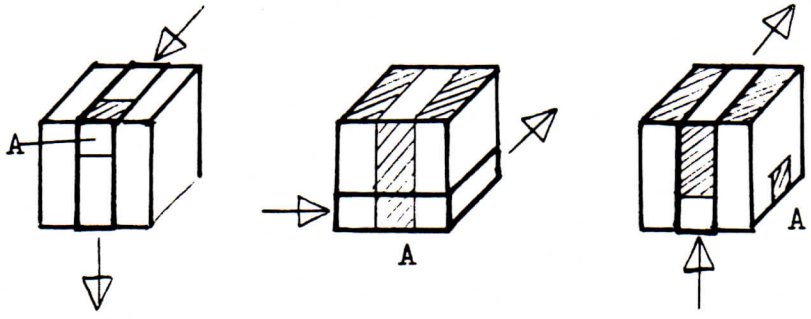
The two cubies to be exchanged are turned into the bottom layer and are positioned one after the other.



The solutions of problem 2d and 2e can be combined to form the solution for problem 2c.

We also can take another path. We can bring the edge cubies into the bottom layer and restore them from there. For example, edge cubie A (Fig. 29) can go into any of the other edge cubie positions.

Fig. 29



Move 1 brings A into the bottom layer. Move 3 is move 1 done backwards and A stays in the bottom layer. We can bring A into position b, c, or d as well (Fig. 3o). In this case, the cubie to be replaced by cubie A has to go into the bottom layer like A did in move 1.

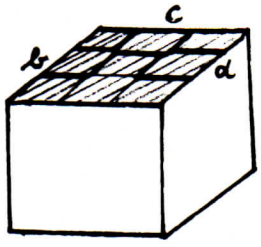


Fig. 3o

In cases b or d, cubie A first must move out of the vertical layer containing cubie b and d. Do this with a quarter turn of the bottom layer. In all three cases move the vertical slice containing the cube you wish to exchange with cubie A (as in move 1). After a quarter move of the bottom cubie A is in its position. A vertical slice move brings it back to its place in the top layer.

So one by one the edge cubies are placed until all of them are in the desired positions.

Alternative B :

If you start with a shuffled cube solving problem 2, you can go the way of alternative A but there is a more elegant and more direct way. When you collect the facelet of one color on the same face you can simultaneously place the cubies into the correct positions relative to each other. Doing this no cubies have to be exchanged later like in way A.

Problem 2f : Take the shuffled cube and restore one face, so that the colors on the facelets of the adjacent four faces in the upper layer are correct. Do not exchange cubies like in way A. Restore the cubies correctly at once.

Solution to problem 2f :

In the beginning you have to think of the order in which the cubies shall be positioned. You can take the order that is used in the solution for problem 1. With the center cubie you determine the color to be restored. Then you position the edge cubies followed by the corner cubies.

The first edge cubie can be moved into an arbitrary position in the top layer (cubie A in Fig. 31a). The positions for the other seven cubies in the chosen color are determined by this.

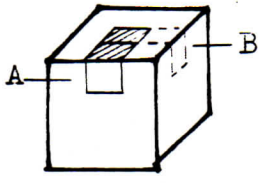


Fig. 31a

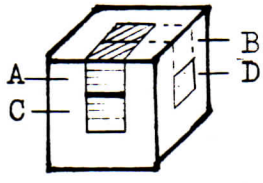


Fig. 31b

Now the second edge cubie shall be positioned. We want to set the position B in Fig. 3o and have to think about which edge cubie will fit here. The color pointing to the top is already determined. So, what is the color for the other facelet ? This question is easily answered. We turn the lower two layers until below A the center cubie C with the same color arrears (Fig. 31b). The second facelet has to show the same color like the center cubie D.

Now we know which edge cubie fits into position B. We look it up on the cube and position it like described in problem 1a to 1c.

Then we turn the lower two layers until C and D are below A and B (Fig. 31b). After this we know how to restore the remaining two edge cubies in the top layer.

Now we can set the corner cubies. The colors of the edge cubies determine exactly what corner cubie will fit. We only have to apply the sequences in problem 1d - 1f.

Now let us approach the problem in a different way. Instead of orienting the top edge cubie first, let us position any cubie in the top layer. Then use its colors to determine which cubies should be its neighbors, and continue in this fashion. That is, use the colors of an edge or corner cubie to determine their neighbors. For example refer to Fig. 32.

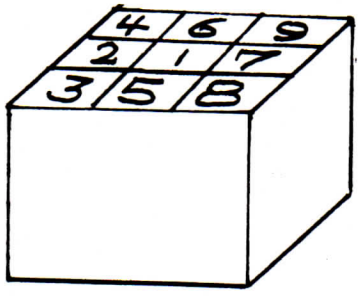


Fig. 32

Notice that next to edge cubie 2 the two neighboring corner cubies 3 and 4 reside. Then two more edge cubies follow : 5 and 6. The neighboring corner cubies 3 and 4 determine which two edge cubies lie next to them. This leaves one remaining edge cubie 7. The final two cubies, cubies 8 and 9, fall into the last two open positions.

Problem 2g : Restore the upper layer using the order given in Fig.32

Solution to problem 2g :

Except for setting edge cubies 5 and 6, everything has already been discussed. Thus there remains only a part of the problem to solve. We want to position cubies 1 through 4 and then determine which cubie belongs in position 5. Before proceeding, let us put it in position A as seen in Fig. 33.

Problem 2h : Bring the edge cubie from position A into position 5 (see Fig. 33).

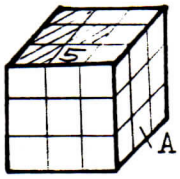
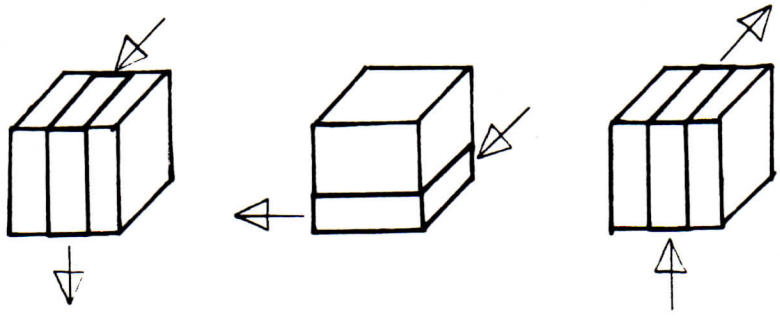


Fig. 33

Solution to problem 2h :

Turn the vertical middle layer containing the top center cubie and the cubie in position 5 toward you. Then by turning the bottom layer, set cubie A to replace the cubie that was brought down from position 5. Now turn the vertical middle layer back, bringing cubie A into position 5.



The edge cubie for position 6 can be placed in a similar manner.

In closing this chapter, some other interesting orders are discussed.

Problem 2i : Order the top layer. However, this time choose a center cubie and arrange the corner cubies first. Then arrange the edge cubies.

Solution to problem 2i :

Most of this problem has already been discussed. We already know how to position the corner cubies. We will now discuss how to position the edge cubies without disturbing the rest of the top layer.

Problem 2j : The corner cubies are already positioned as in Fig. 34. The edge cubie A lies in the middle layer and is to be brought into position B.

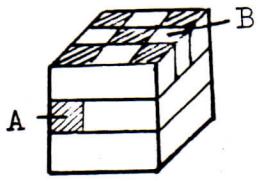


Fig. 34

Solution to problem 2j :

Fig.35 shows a solution with three moves.

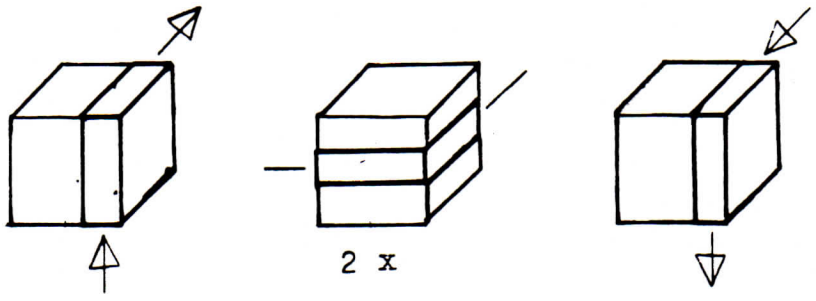


Fig. 35

In summarizing the solutions for the problems of this chapter, we see that we can restore the cubies in the upper layer in an arbitrary manner.

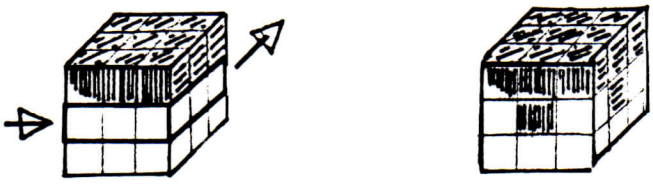
In order to be prepared well for the next chapter, we should thoroughly practice restoring the top layer. As a test, let us reshuffle the cube and restore the top layer without looking at this booklet.

Did you manage it ?
If so, you have taken a major step on the way to completely restoring the cube.

1.3 Restoring the middle layer

We know now how to arrange the top layer. Next we want to position the cubies in the middle layer.
As we have seen in section 1.2, placing the four center cubies in the middle layer is a simple matter. Simply turn the middle layer until any center cubie matches the edge cubie above it in color. Automaticly the other three center cubies in the middle layer fall into their correct position.

Fig. 36 : Turn the middle layer in the direction of the arrow until you achieve the proper orientation.



Now we already have four of the eight cubies in the middle layer positioned correctly. Let us now arrange the four edge cubies, one at a time.

Look for an edge cubie in the bottom layer that fits into the middle layer. We wish to position this cubie correctly without disturbing the cubies in the upper layer. To do this, first turn the bottom layer until the bottom edge cubie to be arranged matches the color of its side facelet with that of the center cubie directly above. This will result in one of two positions which are mirror images of each other (Fig. 37). Notice that five uniformly colored cubies on a side form a large T.

Problem 3 : Bring cubie A into position B without disturbing previously arranged cubies.

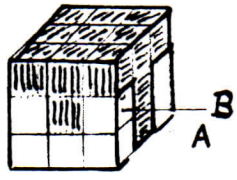


Fig. 37a

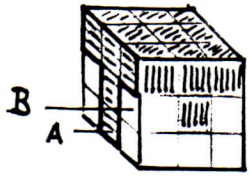


Fig. 37b.

Warning ! This is a difficult problem !

If you wish to simplify it, use the hints contained in the six numbered sentences.

The first hint in itself helps a great deal and is followed by five others that make the problem progressively easier.

In this way you can determine the appropriate or suitable degree of difficulty of this task.

Refer to Fig. 37a for the hints given below. In case the starting position in Fig. 37b applies, replace "right" with "left" in the sentences below.

Hints :

1. The solution lies in eleven moves. These consist only moves of the right and lower layer.

2. The eleven move sequence is symmetrical. After five moves of the lower and right layers, move the bottom layer, and finish with the first five moves done in opposite order and in opposite direction.

3. Start with the right layer, and turn in alternate fashion the lower, right, lower, ... layers.

4. Only 90° degree turns (quarter turns) are used.

5. Moves in the right layer are done in alternating directions.

6. The first two moves in the bottom layer (that is moves 2 and 4) point in the same direction.

Solution to Problem 3 :

The hints on the previous page outline the solution. If you use all six hints, you need only work out all the remaining possible sequences. We have two possible directions in each of the first, second and sixth moves. The remaining moves are determined by symmetry and the other hints. Thus 2 x 2 x 2 = 8 different sequences remain, each of which can be tested easily.

The sequence in Fig. 38a demonstrates the exact solution for the starting position of Fig. 37a.

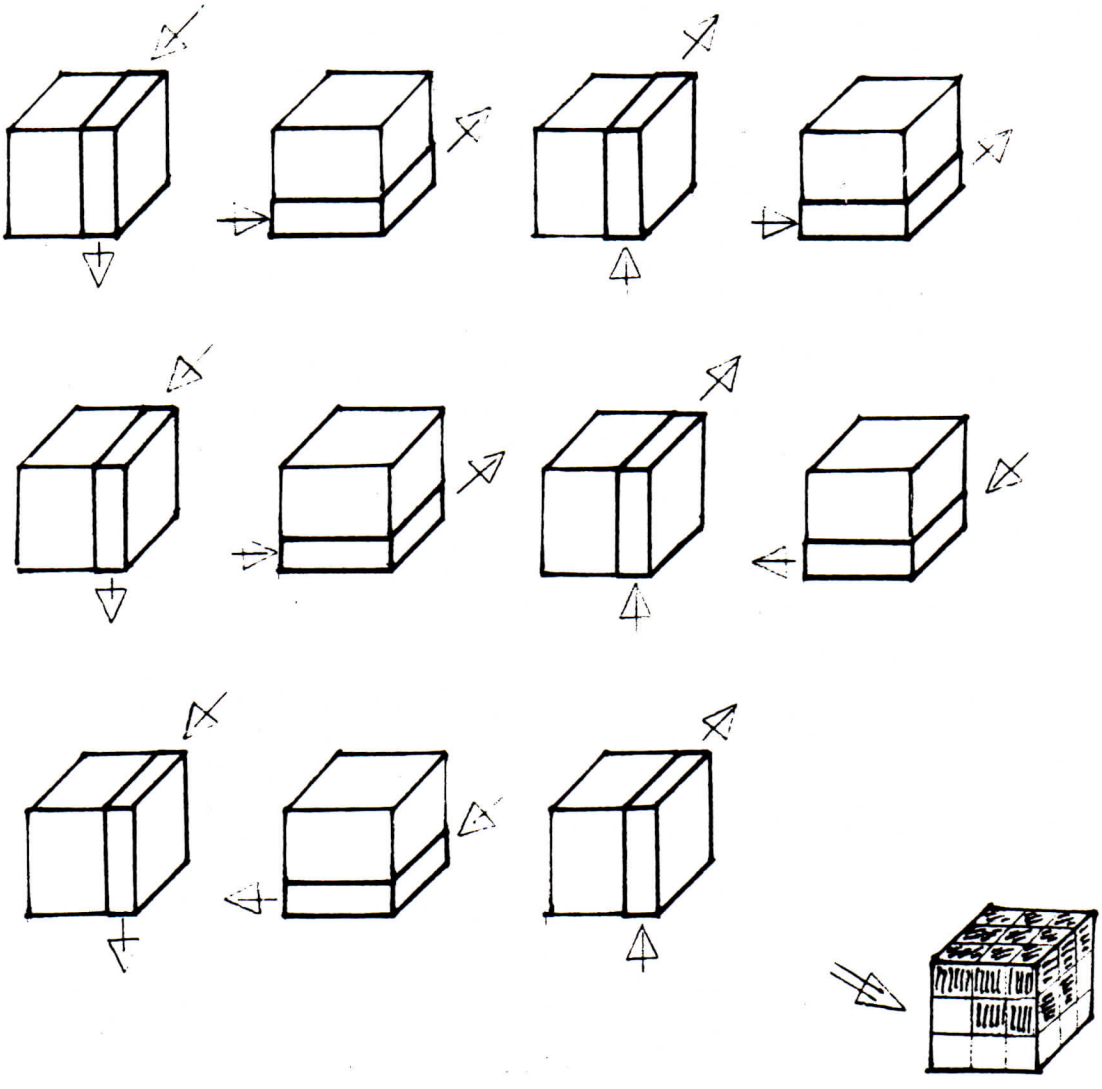
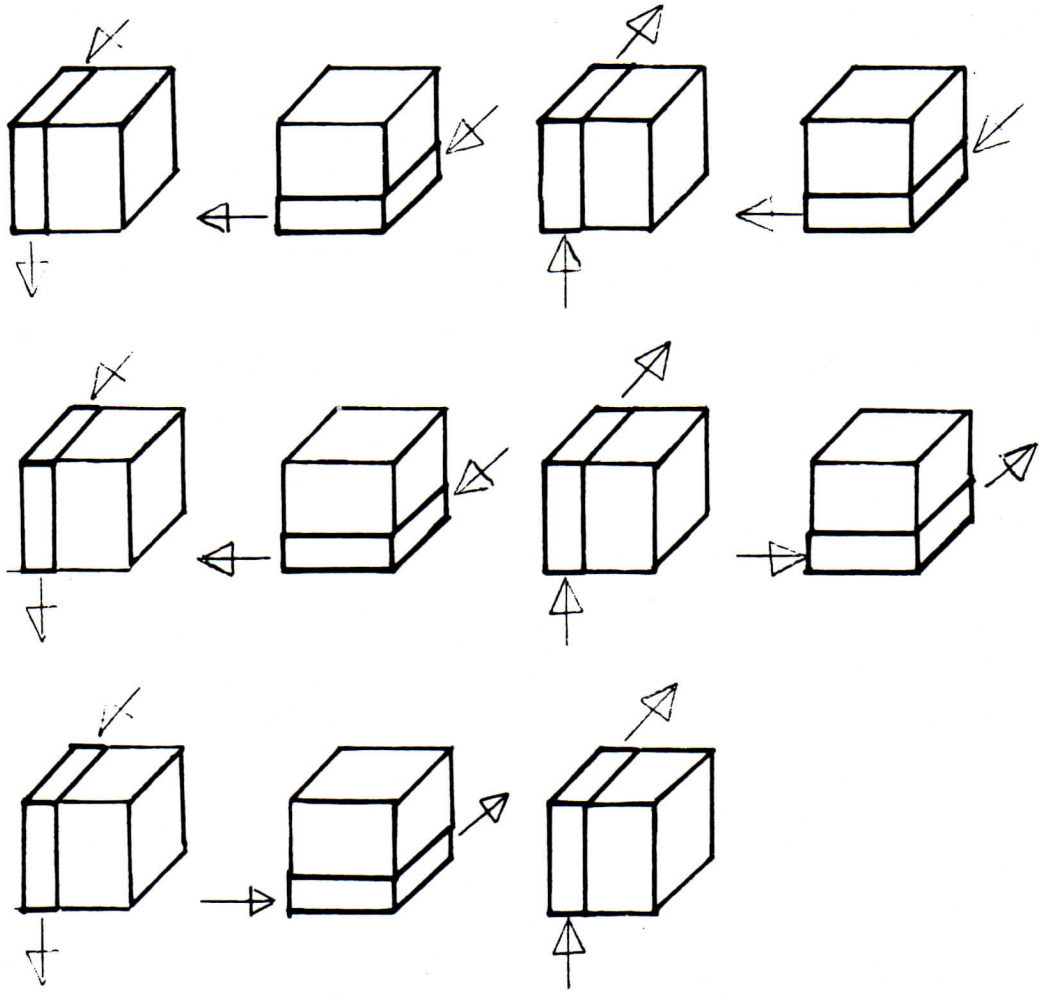


Fig. 38a : 11-step
right

If the starting position of the cube corresponds to Fig. 37b, the moves are done on the left and lower layers instead; they are mirror images of the steps in Fig. 38a.

Fig. 38b : 11-step left



You can hold the cube with the left hand to turn the right layer or with the right hand to turn the left layer (see Fig. 39. In either case, the following rule of thumb applies :

Rule of thumb :

First apply three moves on the bottom layer in the direction of the thumb. The last two movements of the bottom layer point against the direction of the thumb.

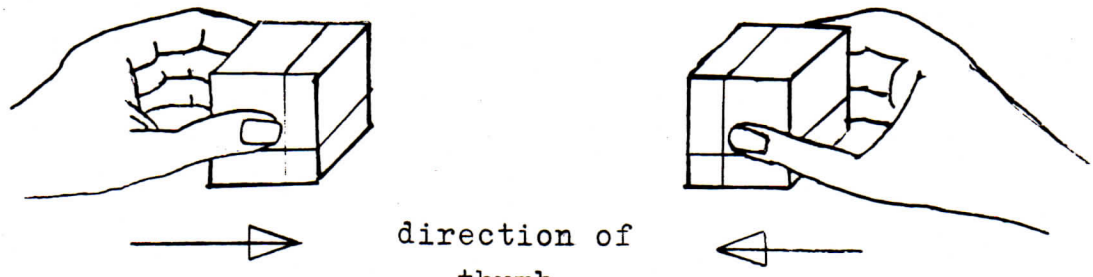


Fig. 39

Now we know how we can bring the edge cubies from position A to position B in figures 37a and 37b.

If the edge cubie is already in the middle layer but in the wrong orientation, bring it into the lower layer by one of the two 11-step sequences. From there it can be oriented correctly by building the T and applying the correct 11-step.

By this way, for example, you can exchange the two edge cubies A and B in Fig. 4o.

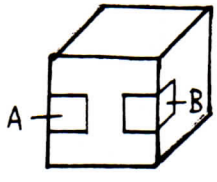


Fig. 4o

These two 11-step sequences enable us to arrange the whole middle layer.

2. Restoring the bottom layer

Now we have already restored the two upper layers. The center cubie on the bottom is already correct. Thus only eight other cubies remain to be positioned, and we must maintain the two upper layers during all subsequent movements.

We can arrange the bottom layer in several steps. First, the bottom face should be arranged without looking at the adjacent side facelets in the lower layer. After that, the side facelets are arranged correctly, and we are done.

2.1 The cross

In trying to get a uniformly colored bottom face, we might encounter the cross pattern on the bottom face (Fig. 41). The bottom facelets of all four bottom edge cubies match the color of the bottom center cubie.

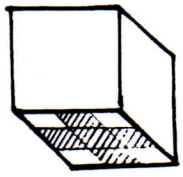


Fig. 41

The cross

After ordering the upper two layers, the bottom can have the patterns given in Fig. 42: the cross, the beam, the angle or the point. The cross is the most important pattern. The other three patterns will be discussed interms of getting to the cross pattern. First, let us examine them.

cross beam angle point

Fig. 42

The beam can be vertical or horizontal. The angle can lie in the upper right, the lower right, the upper left, or the lower left.

In order todescribe the bottom. let us tilt the cube 90° to the back as in Fig. 43.

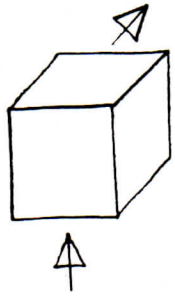


Fig. 43

Then we are able to describe the bottom as we see it in front of us. The drawing of the bottom face in Fig. 44a corresponds to the three-dimensional drawing in Fig. 44.

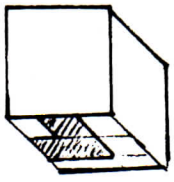


Fig. 44

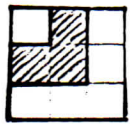


Fig. 44a

If you don't have the cross on the bottom now, you have the starting position for one of the next problems.

Problem 4a : Make a cross out of the vertical beam in Fig. 45a.

Problem 4b : Make a cross out of the angle in the upper left in Fig. 45b.

Problem 4c : Make an angle out of the point in Fig. 45c.

Problem 4d : Make a beam out of the point in Fig. 45c.

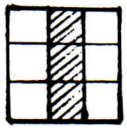


Fig. 45a

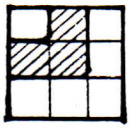


Fig. 45b

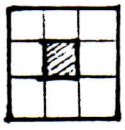


Fig. 45c

Again, several hints are offered. With these you can determine the degree of difficulty of the task on your own.
The hints apply to all four problems.

Hints :

1. A sequence will solve the problem in six moves.
2. The moves of this sequence are done on three layers.
3. These layers are right (r), back (b), and down (d).
4. In every layer, two moves are done, one in both directions.
5. Only quarter moves are used.
6. The order of the layers is :

   for problem a and c : r, b, d, b, d, r ;

   for problem b and d : r, d, b, d, b, r .

Solutions to problems 4a to 4d :

If you use all the hints, there remain to be worked out two possibilities for each of the first three moves : that is, altogether eight possibilities. The following diagrams show the sequences that do solve the problems.

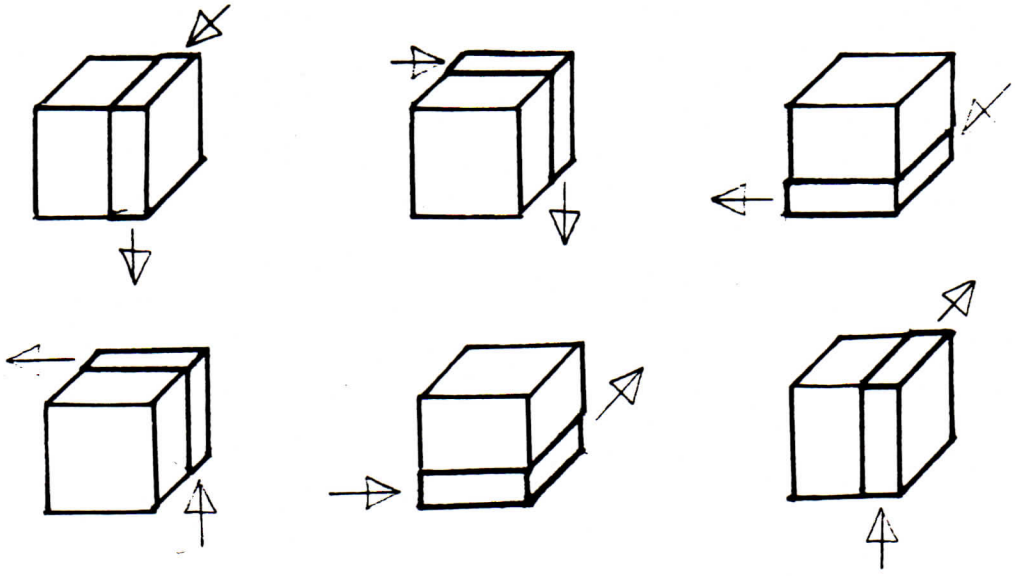


Fig. 46 : Solution to problems 4a and 4c - 6-step forward

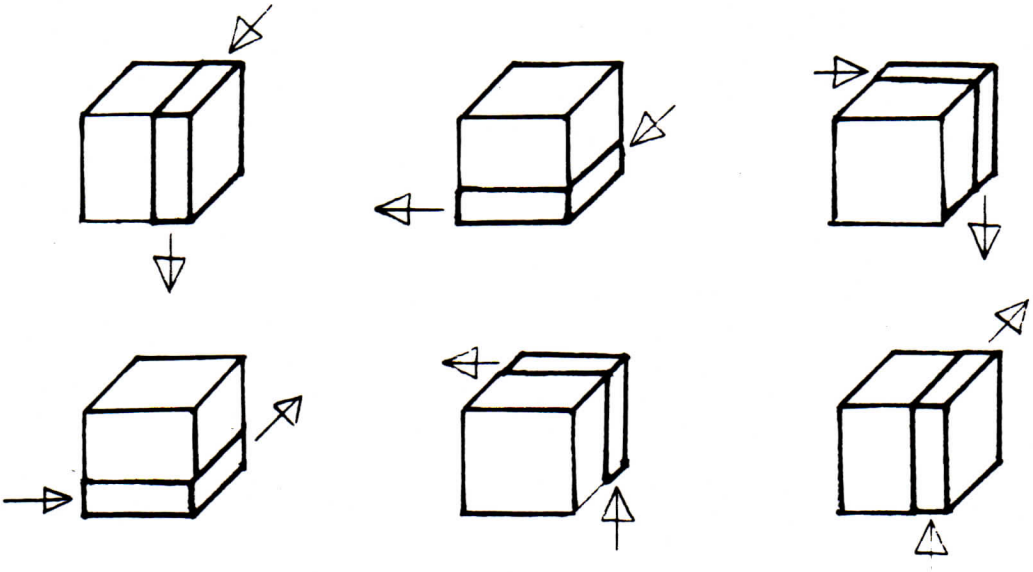


Fig. 47 : Solution to problems 4b and 4d - 6-step backward

Both sequences have an opposite order in the moves applied on the back and lower layers. Also, one version is the backward version of the other : moves are done in opposite order and direction.

In order to transform a point into a cross, you can do so indirectly by passing through the beam pattern, or by passing through the angle pattern. In using the sequence of Fig.46 or 47 orient the whole cube so that the beam is vertical or so that the angle is in the upper left of the bottom face. Then apply the sequence.

2.2 Restoring the bottom face

The objective in this chapter is to color the bottom face uniformly. At this point, both upper layers are restored and there is a cross at the bottom. Now our goal is to arrange the four bottom corner cubies so that the bottom facelet of each corner cubie matches the bottom center cubie. We do so by twisting and exchanging these cubies.

Eight different possible patterns exist for the corner cubies seen from the bottom. For practice, let us enumerate these patterns by a sequence that rearranges the four corner cubies at the bottom without disturbing anything else. This will allow us to track through all the possible patterns.

Problem 5a : Find a sequence which changes the four bottom corner cubies without disturbing the rest of the cube.

Hints :

1. There exists a solution of seven moves.
2. All moves for the right handed solution are done on the right and lower layers.
3. Moves of the sequence are done alternately on the right and lower layers.
4. The right layer is turned alternately in opposite directions.
5. The bottom makes a total turn of 360°.

Solution to problem 5a :

See Fig. 48.

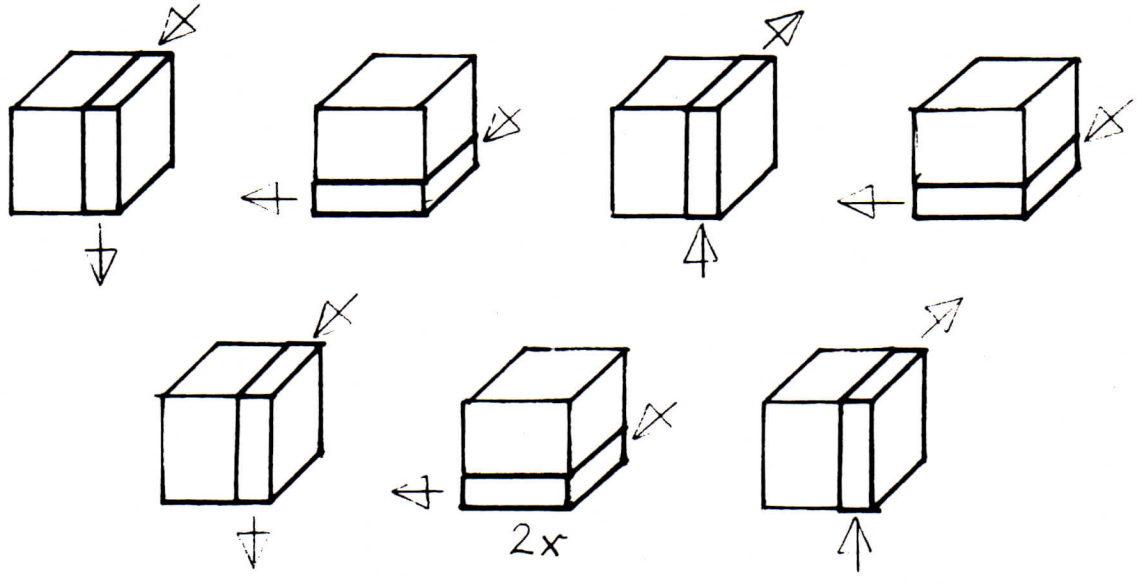


Fig. 48 : simple 7-step right

In Fig. 49 the corner cubie in position A walks through positions A, B, C, D, E, B, A.

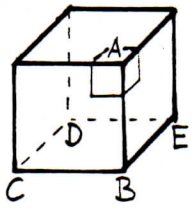


Fig. 49

The corner and edge cubies on the bottom exchange as in Fig. 5o

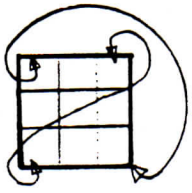
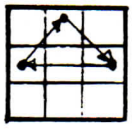


Fig. 5o

Bottom view



The corner cubies exchange in pairs across the two diagonals; all corner cubies except the lower right one experience a counterclockwise turn. In addition the three lower edge cubies rotate clockwise so as to maintain the cross pattern.

This sequence also has its left-handed version : simply replace a clockwise turn of the right with a counterclockwise turn of left layer and vice versa.

With these two sequences (left- and right-handed), we can start from any cross pattern and go to the uniformly colored bottom face.

Problem 5b : Apply the sequence we have learned to find all other possible patterns with a cross in the bottom face. Apply both sequences (left or right) on the previously found cross patterns on the bottom.

Hints :

1. Eight different patterns exist. Patterns that are related by a turn of the whole cube count as one pattern.

2. Turn the already found patterns by a quarter, half or three quarter move of the bottom layer, and apply one of the 7-step sequences.

3. Pay attention to the facelets adjacent to the bottom face which have the same colors as that of the bottom center face.

Solution to problem 5b :

The eight different patterns are :

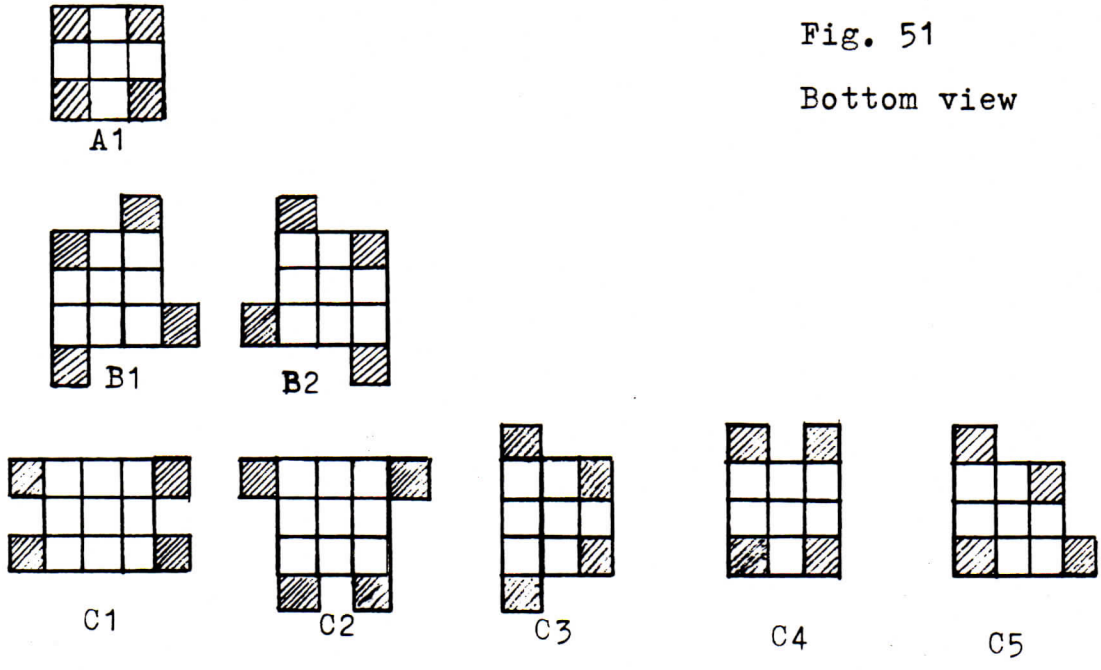


Fig. 51

Bottom view

The colored squares indicate facelets which match the colors of the bottom center cubie. Squares attached to the 3x3 bottom face indicate facelets adjacent to the bottom face.

C1 and C2 share the same pattern on the bottom face and differ only in adjacent facelets. A1 is the uniformly colored bottom face which is the goal of this chapter. B1 and B2 are the two patterns that can be obtained with one 7-step sequence.

The patterns in the C-row can transform into one of the patterns in the B-row by applying a 7-step sequence.

Patterns such as 51a are only theoretically possible; they cannot be obtained for a configuration of cubies which allows all four to be one color (our goal).

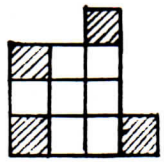


Fig. 51a

In chapter II.1, we show in fact that only a twelfth of all imaginable patterns of the cube are obtained by ordinary moves. Only if you physically take the whole cube apart and assemble it differently is the pattern shown above accesible. But then you would no longer be able to obtain the completely ordered cube.

Now let us see how to get the A pattern by applying 7-steps on a B or C pattern.

Problem 5c : Transform the B and C pattern into the A pattern with (the least possible) 7-steps.

Hints :

1. Do the 7-step backward, and try to obtain the B and C patterns from the A pattern.

2. The B (C) patterns in Fig. 51 are already turned in such a way that one of the two 7-steps (left or right) can result in the desired A (B) pattern.

3. In the process of transforming the C pattern into the A pattern, we pass through a B pattern after applying one 7-step and an extra turn of the bottom. This B pattern can then be directly transformed into an A pattern.

Solution to problem 5c :

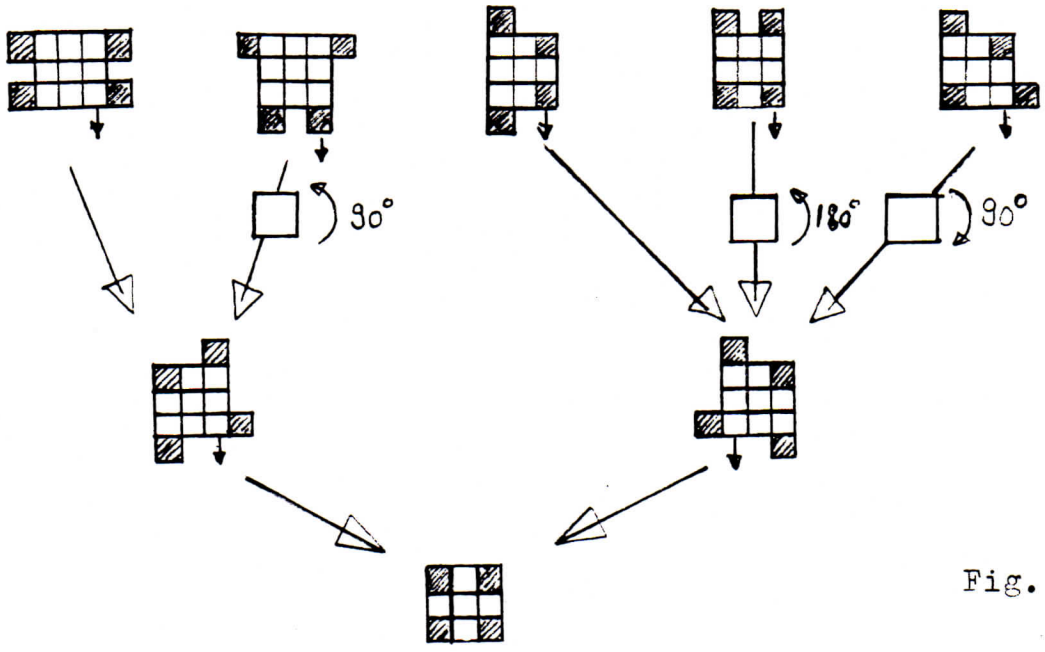
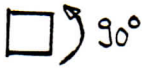


Fig. 52

In Fig. 52 and 53, the little arrow pointing to the right or left means apply the 7-step right or left.

Fig. 53



means that the whole cube should make a quarter turn counterclockwise on its vertical axis.

Thus we can color the bottom face uniformly, starting from any arbitrary pattern.

2.3 Restoring facelets next to the bottom

Our final goal is just within reach. We only need to exchange bottom corner cubies, followed by edge cubies in order to arrange the adjacent facelets.

2.3.1 Exchanging corner cubies

We have a problem similar to that in the beginning of chapter 1.2, alternative A : Two corner cubies are to be exchanged over a diagonal or over an edge.

The problem at hand is more complicated, however : now we want to keep the upper two layers and the bottom face in order. But by now we have more experience with the cube, and this should not present us with too formidable a problem.

Exchanges across an edge occur more often, so let us handle them first. If an exchange along a diagonal is necessary on your cube now, continue with problem 6b.

Problem 6a : Exchange two corner cubies in the lower layer along an edge.

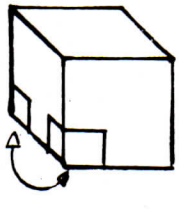


Fig. 54 :

Keep the two upper layers and the bottom face undisturbed !

Hints :

1. You can obtain pattern B2 in fig. 51 in seven moves. With an additional simple 7-step left we have obtained the desired exchange.

2. This new 7-step that we want to call double-sided 7-step is done in the right, the left and in the lower layer.

3. The sequence of the layers to be turned is :
   r, d, l, d, r, d, l.

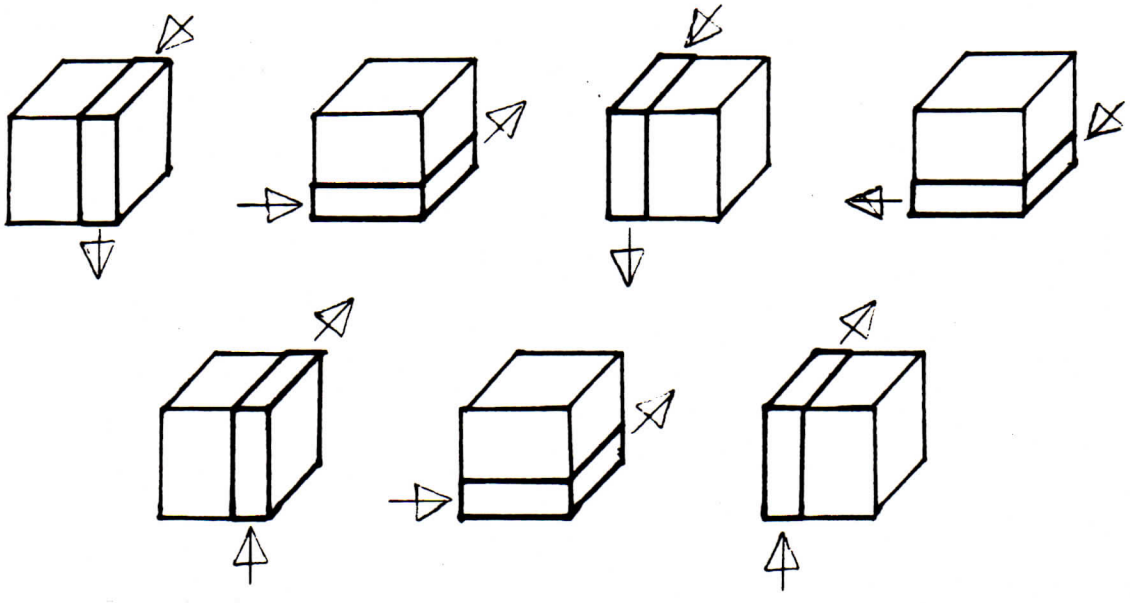
4. There are only quarter moves.

5. One of the two moves in the right (left) is clockwise, the other is counterclockwise.

Solution to problem 6a :

The double-sided 7-step is shown in Fig. 55 .

Fig. 55 :



Now we have the bottom pattern B2 from Fig. 51 and apply the simple 7-step left. The bottom is now uniformly colored again and the corner cubies in the bottom layer are all in the correct positions.

Problem 6b : Exchange two corner cubies in the lower layer along the diagonal in between (Fig. 56). Keep both upper layers and the bottom face in order.

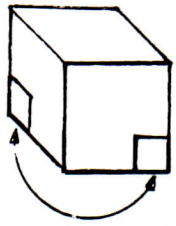


Fig. 56

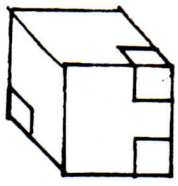


Fig. 57

Hints :

1. After ten moves you can obtain an angle at the bottom. With a 6-step backwards from Fig. 57 you will get pattern B1 in Fig. 51 and from there you can procede with a simple 7-step right and obtain the solution to the problem.

2. In this 1o-step you have to turn the right, back, left and lower layer.

3. The 1o-step is very similar to the 6-step backwards.

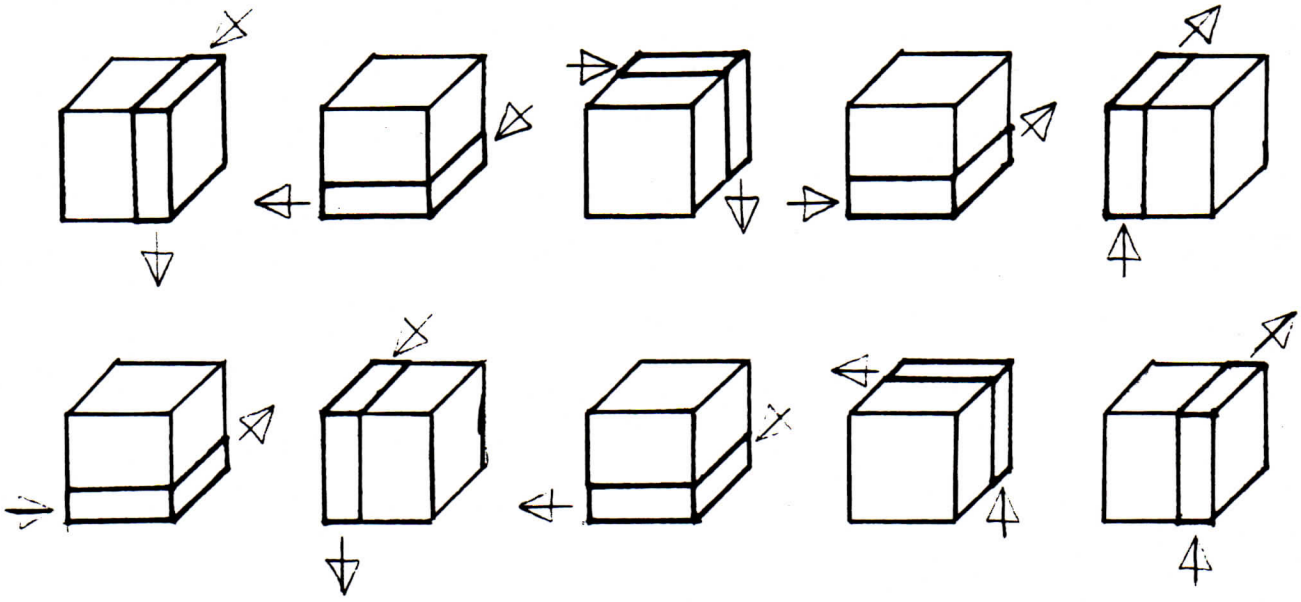
4. The sequence of the layers to be turned is :
   r, d, b, d, l, d, l, d, b, r.

5. The cubie A in Fig. 57 turns down to the lower layer with the first move and returns with the last move. It is never in the left or back layer when it is moved.

6. There are only quarter moves.

Solution to problem 6b :

The 1o-step you are looking for is : Fig. 58



The cubie A in Fig. 59 makes the following walk :

A - B - C - B - D - B - A

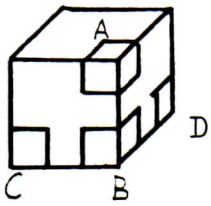


Fig. 59

After the 1o-step you cannot immediately see that it solves the problem. A turn of the whole cube brings the angle at the bottom to the upper left. We apply the 6-step backwards in Fig. 47 and obtain the pattern B1 after another turn of the whole cube. Then we continue with the 7-step right (in Fig. 48).

Now the bottom face is uniformly colored again and the corner cubies in the lower layer are correct.

From chapter 1.2 we know, that with the exchange along an edge or a diagonal we can rearrange an arbitrary constellation of corner cubies.

Now we only have to arrange the edge cubies at the bottom layer and then the cube is completely restored.

2.3.2 Exchange of edge cubies

Anybody who has arrived at this chapter can do the last steps easily.

After the upper two layers, the bottom face and the bottom corner cubies are in order, there are only four different positions left to deal with. Fig. 6o describes the completely restored bottom layer.

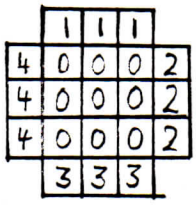
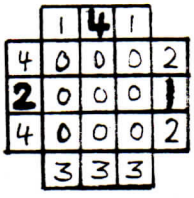


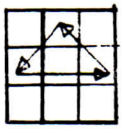
Fig. 6o

Every number symbolizes a color.

The four different patterns to deal with are :

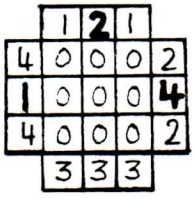
Fig. 61a

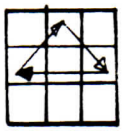




exchange counter-clockwise

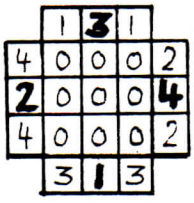
Fig. 61b

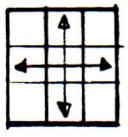




exchange clockwise

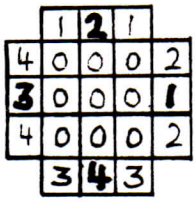
Fig. 61c

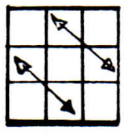




exchange vertically & horizontally

Fig. 61d





exchange along the side diagonals

In Fig. 61 a and b there are three in Fig. 61 c and d four edge cubies to be restored.

Again look for the problem that is on your cube now.

Problem 7a : Exchange three edge cubies counterclockwise (Fig. 61a).

Problem 7b : Exchange three edge cubies clockwise (Fig. 61b).

Problem 7c : Exchange two pairs of edge cubies along the vertical and horizontal (Fig. 61c).

Problem 7d : Exchange two pairs of edge cubies along the side diagonals (Fig. 61d).

General Hint :

These problems can all be solved with several simple 7-steps left and right (Fig. 48), which are done one after the other.

Hint for 7a and 7b :

1. You can solve the first two problems by two simple 7-steps, one right and one left each.
2. In the beginning of every 7-step sequence the same corner cubie is brought from the upper into the lower layer.

Hint for 7c and 7d :

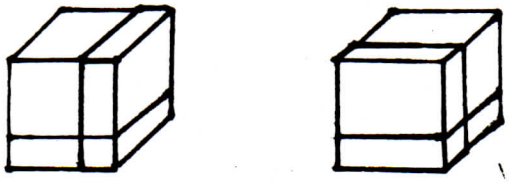
1. The last two problems can be solved by four simple 7-steps each.

2. The first two and last two sequences are of the same type each. This means both are right-sided or both are left-sided 7-steps, and are done in the same layers.

Special hint for 7c :

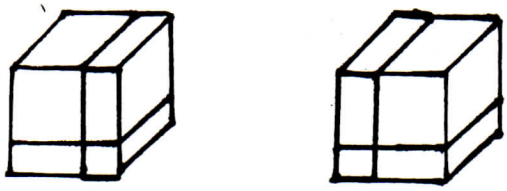
3a. All four sequences are of the same type. The last two sequences are done in the layer which is left adjacent to the layer for the first two moves (Fig. 62).

Fig. 62



Layers for the two first and last 7-steps.

Fig. 63



Special hint for 7d :

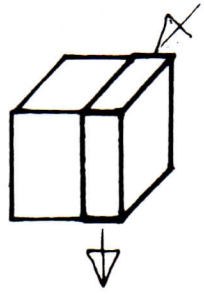
3b. The two last 7-steps are done backwards. Fig. 63 shows the layers to be turned.

Solution to problem 7 :

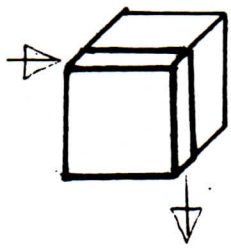
We turn the whole cube so that the bottom is in the position of Fig. 61a, 61b, 61c or 61d. In describing the solutions of the problems it is sufficient to show the first move of the simple 7-steps and its kind - left or right.

Solution to problem 7a :

Three edge cubies are exchanged counterclockwise.



1st sequence : simple 7-step right

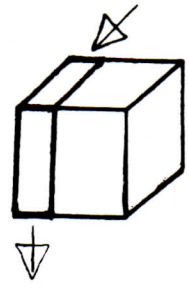


2nd sequence : simple 7-step left

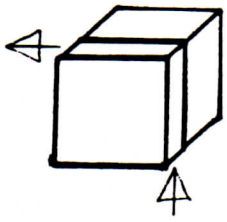
Fig. 64

Solution to problem 7b :

Three edge cubies are exchanged clockwise.



1st sequence : simple 7-step left



2nd sequence : simple 7-step right

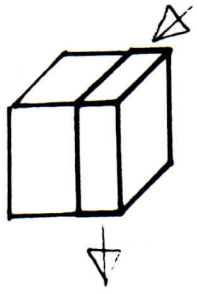
Fig. 65

In order to exchange counterclockwise we start with a simple 7-step right and in order to exchange clockwise with a simple 7-step left.

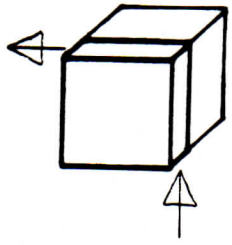
Solution to problem 7c :

The four edge cubies shall be exchanged pairwise over the vertical and horizontal. We need not turn the bottom face into a special position.

As a solution you use four simple 7-steps right: the first two in the right and the bottom layer and the last two in the front and the bottom layer.



2 times simple 7-step right



2 times simple 7-step right

Fig. 66

After the first two sequences you can turn the cube by 9o° to the right like in Fig. 67 and continue with the last two sequences in the right and bottom layer.

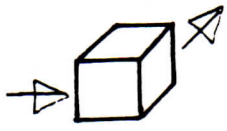
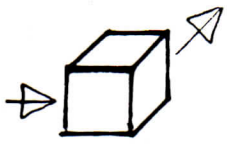


Fig. 67

Solution to problem 7d :

Four edge cubies shall be exchanged pairwise over the side diagonals and we have already turned the cube so that the bottom face shows the pattern from Fig. 61d.

First there are two simple 7-steps right in the right layer. Then the cube is turned by 18o° like in Fig. 68.



2 x

Fig. 68

Now two simple 7-steps right backwards are done in the right layer. Fig. 69 shows one of these sequences backwards.

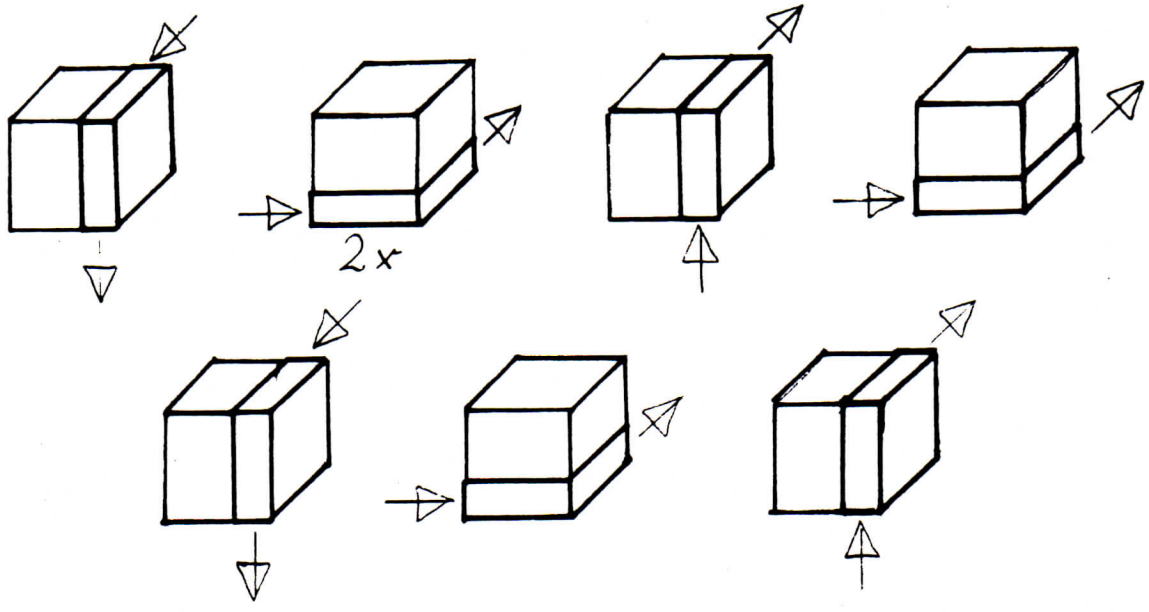


Fig. 69 : 7-step right backwards

The complete circle of the bottom ($180^{o} + 90^{o} + 90^{o}$) is done in the opposite direction than in the forward sequence.

Congratulations, you have reached your goal. You are able to rearrange the completely suffled cube on your own.

To obtain a good survey of the whole solution system in the next chapter all sequences are summarized and explained briefly.

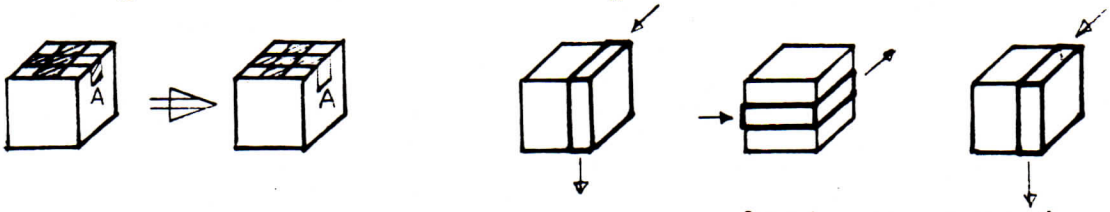
3. Summary of the solution system

In head words and with figures the moves needed are outlined for the whole way of the solution.

Choose the color for the upper center facelet of the cube !

Rearrange the upper layer :

- Exchange the facelets of the edge cubie A :



basic sequence 1

- A in Fig. a goes to B in Fig. b.

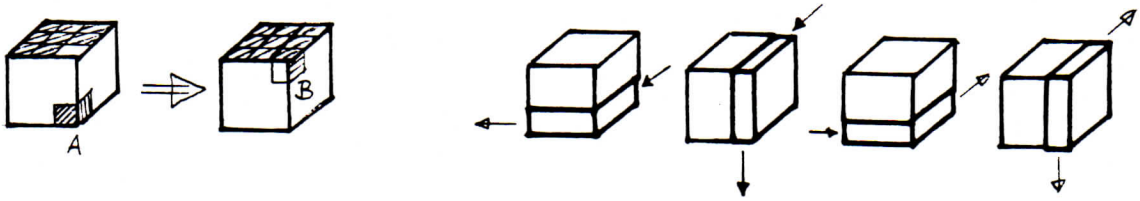


Fig. a Fig. b basic sequence 2

Rearrange the middle layer :

- Turn the four center cubies correctly,
- then rearrange the four edge cubies : C in Fig. c goes to D in Fig. d

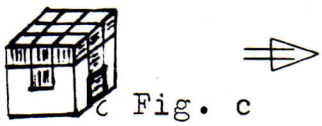


Fig. c

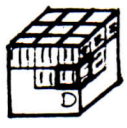
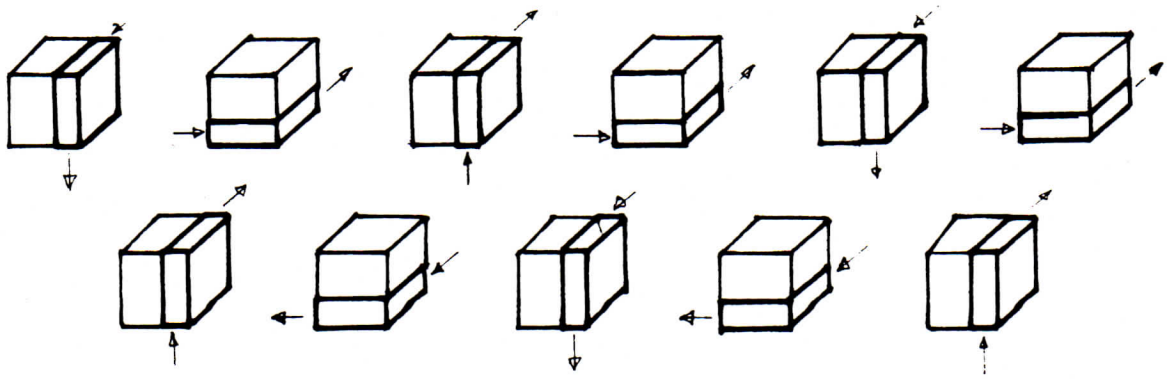
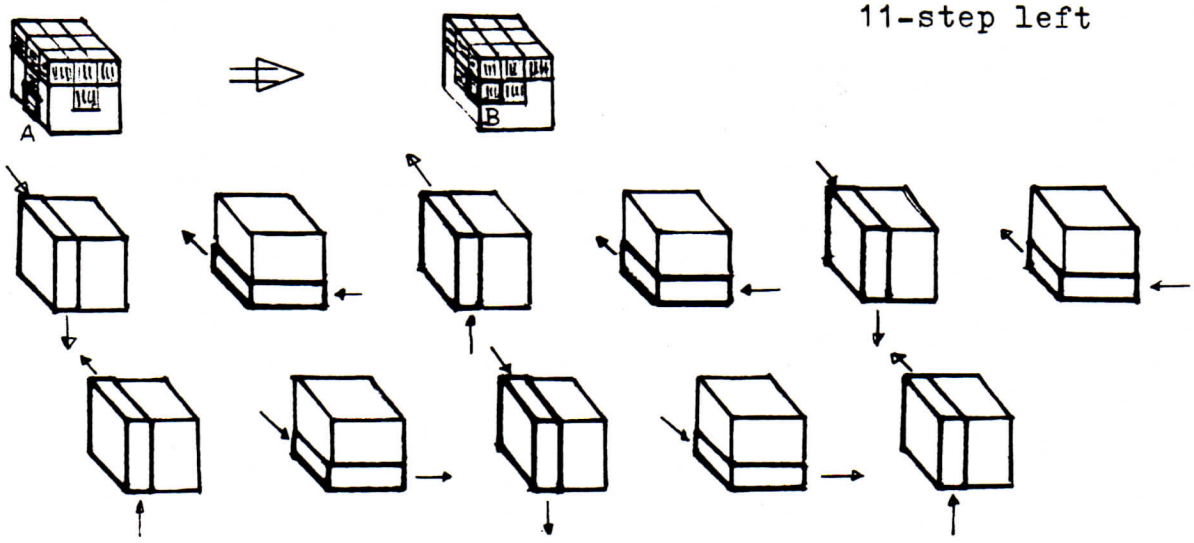


Fig. d

11-step right



11-step left



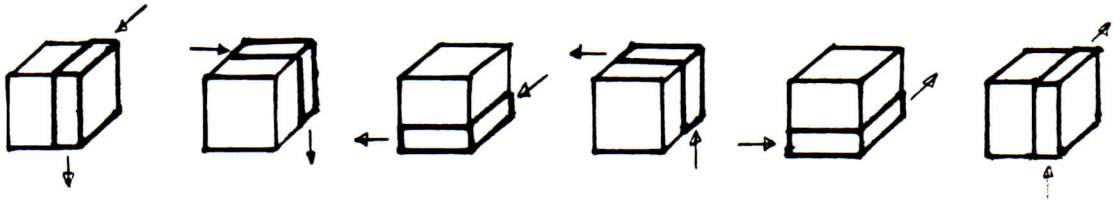
Watch the rule of thumb :
The first three moves in the bottom layer into the direction of the thumb, the last two moves in the opposite direction.

Rearrange the bottom layer :

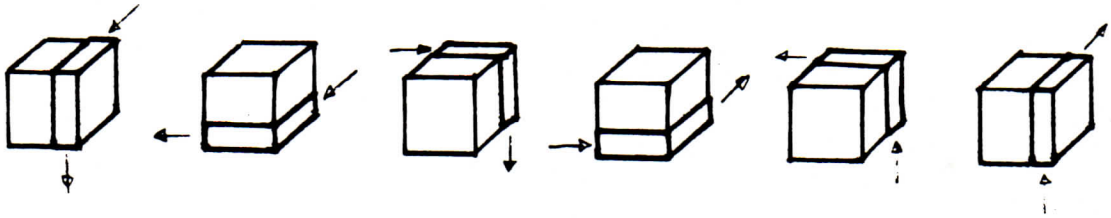
- Create the cross :

possible starting positions of the bottom face :

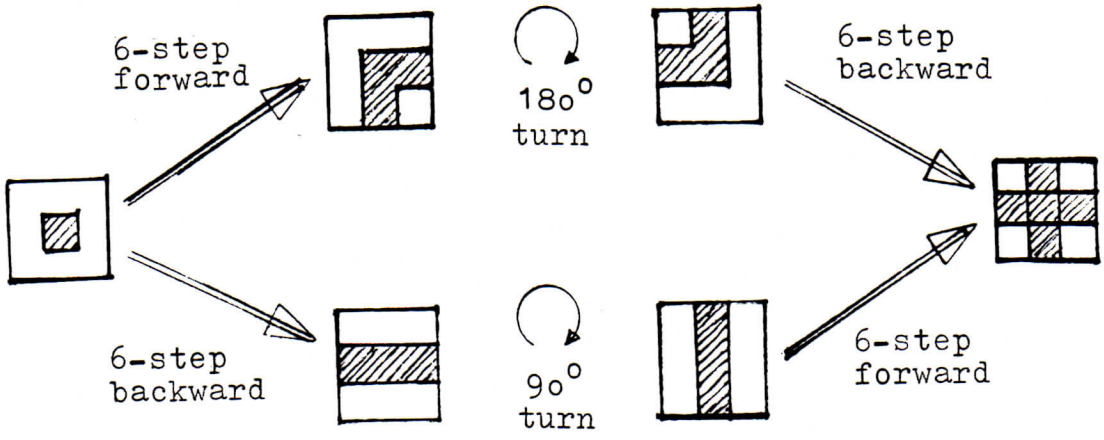
6-step forward :



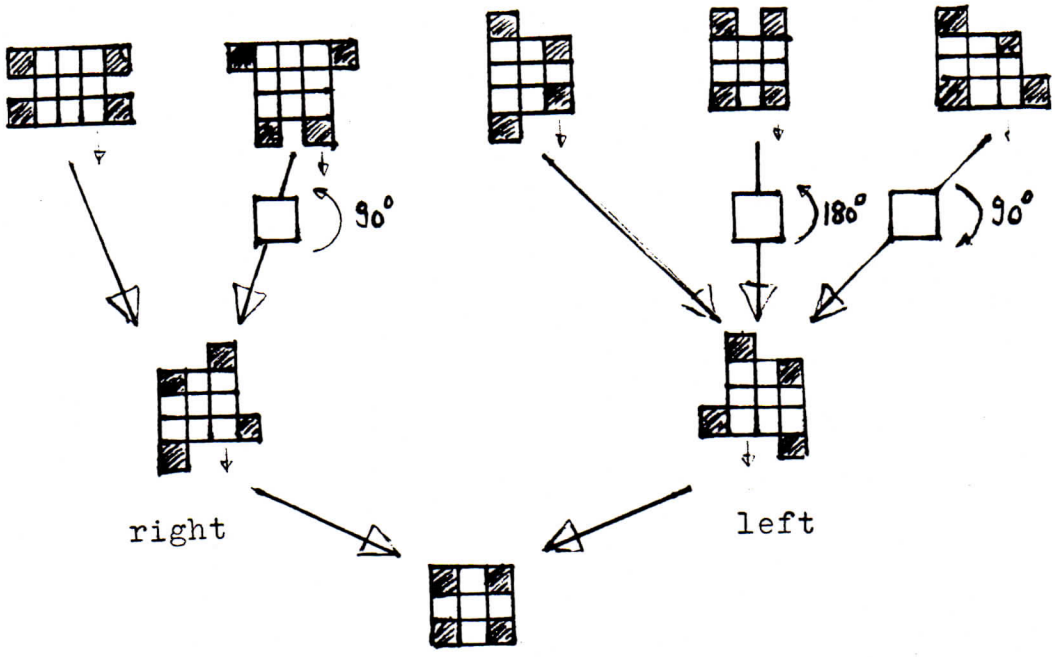
6-step backward :



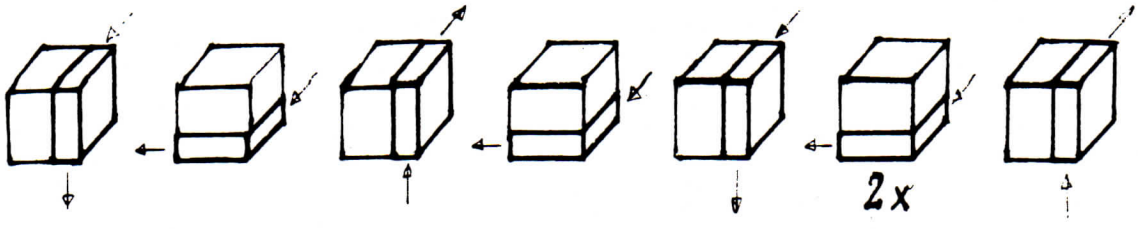
We can transform the point into the angle or beam and this into the cross :



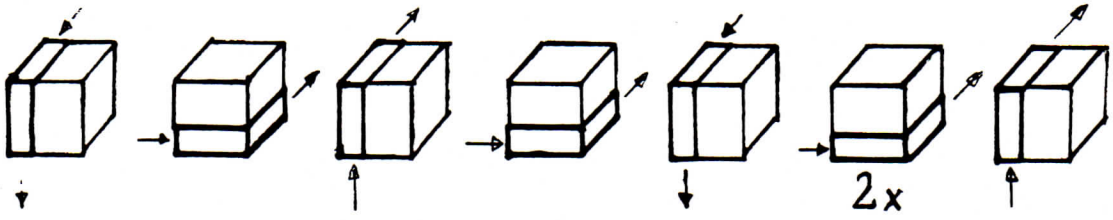
Rearrange the bottom face completely by 7-steps :



simple 7-step right

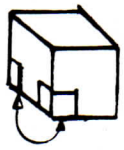


simple 7-step left :

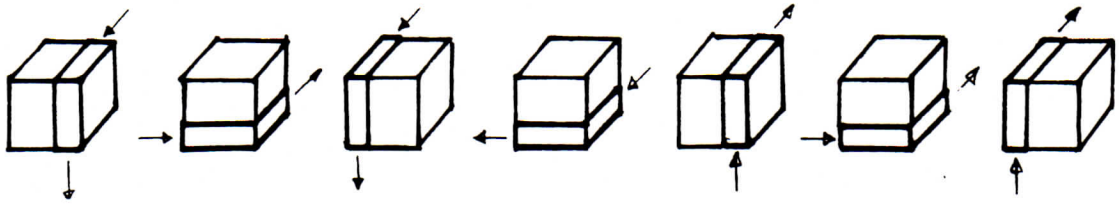


Rearrange the corner cubies in the lower layer:

- if necessary, exchange two corner cubies along an edge :

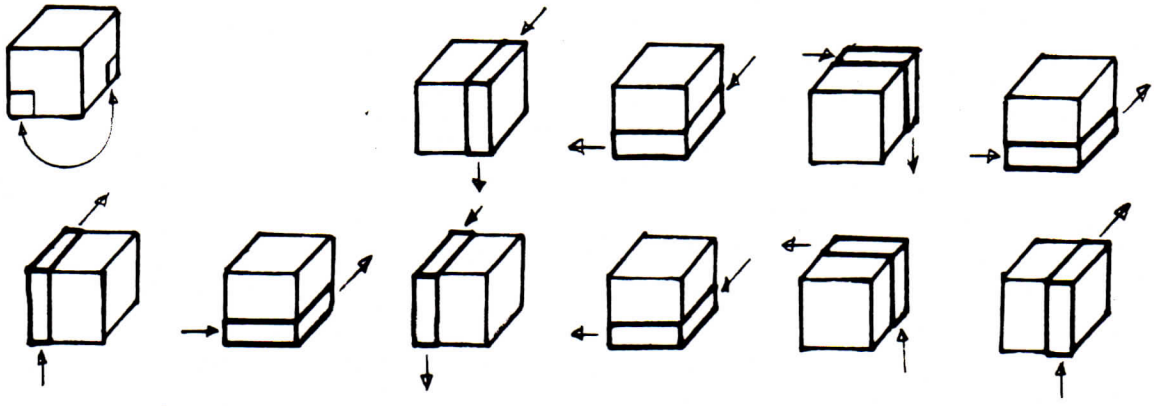


double-sided 7-step



Then a simple 7-step left follows to make the bottom uniformly colored again.

- if necessary, exchange two corner cubies along a diagonal :



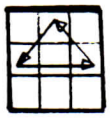
1o-step

Then a 90° turn of the whole cube to the left. After this the angle at the bottom face is in the upper left. Now a 6-step backwards and a simple 7-step right follow and the corner cubies are in place.

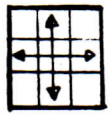
Rearrange the edge cubies in the lower layer :

- Four different exchanges are possible :

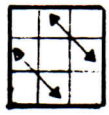
clockwise



counter-clockwise



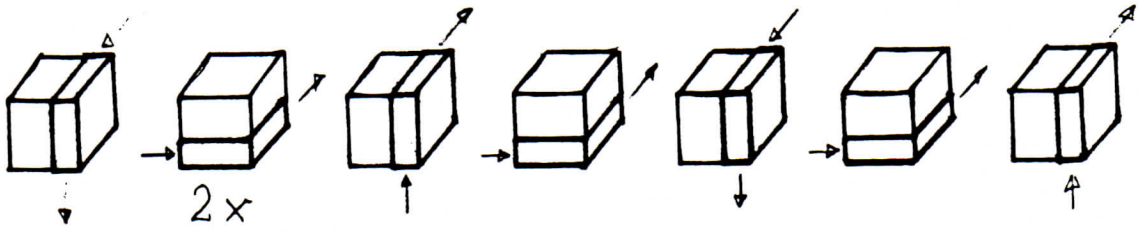
vertical & horizontal



along side-diagonals

- clockwise :
  A simple 7-step left, 90°- turn of the whole cube to the right, then a simple 7-step right.
- counterclockwise :
  A simple 7-step right, 90°- turn of the whole cube to the left, then a simple 7-step left.
- vertical and horizontal :
  Two simple 7-steps right, 90°- turn of the whole cube to the right, then two simple 7-steps right.
- along the side diagonals :
  Two simple 7-steps right, 180°- turn of the whole cube, then two simple 7-steps right backwards.

simple 7-step right backwards :



Altogether eleven sequences are used. Some of them are very similar :

- basic sequences 1 and 2
- 6-step forward and backward
- simple 7-step right and left
- simple 7-step right backward
- double-sided 7-step
- 1o-step
- 11-step left and right.

II. More about the cube

1. The mechanics of the cube

Dealing with the cube you probably have asked yourself : How does the cube work ? How can it be moved in so many different ways and still sticks together ?

These questions you can easily answer on your own, looking into the cube. Remove some of the cubies !

To do this turn the upper layer by 45° like in Fig. 7o, and grasp with the thumb or a flat key between edge cubie A and the middle layer. Press against A until it comes off. Don't worry, you cannot destroy the cube by this.

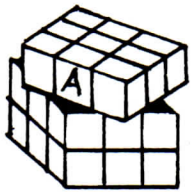


Fig. 7o

Now the other edge and corner cubies can be removed easily. What is left are the six center cubies and the mechanics in between.

Every center cubie can turn arbitrarely around his axis along a screw. After removing a cap of a center cubie with a knife you can see this screw and fasten it if it is loose.

Installing the lower layer we see that the corner and edge cubies have their tracks in which they are moved. They need not be connected firmly to let the cubies stay together.

In restalling the cube you have to watch that it is rearranged in the same way it is taken apart.

If you put it together differently, it may be that by ordinary moves the restored position cannot be reached. The reason for this is explained in the next chapter.

If you have taken apart the cube completely, the most simple way to restall the cube is to allow only one color on a face. So you obtain the starting position directly.

2. Many faces of the cube

Now you have probably handled the cube for many hours and have seen its many patterns. You may have asked yourself : "How many patterns are there ?" We want to answer this question now.

Taking the cube apart we could see that the positions of the center cubies are fixed relatively to each other.

During the assembly we can place an arbitrary edge cubie at an arbitrary edge and an arbitrary corner cubie at an arbitrary corner.

Besides this every edge cubie on every edge has two different positions (see Fig.71) and every corner cubie three on every corner (see Fig. 72). You can flip or twist the facelets of the cubies in each of its possible positions.

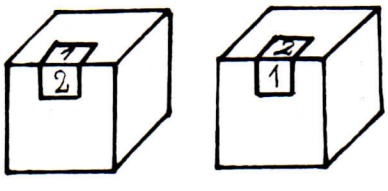


Fig. 71

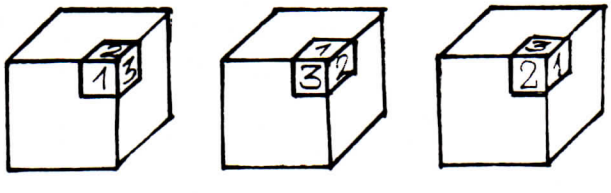


Fig. 72

Assembling the first edge cubie we have twelve edges with two positions each to choose from. For the second edge cube we have eleven edges with two positions each being still free. With every cubie assembled, the number of free places declines. For all edge cubies you receive

$$12 \times 11 \times 1o \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times$$
$$\times \quad 2^{12} = \quad 12\,! \quad \times \quad 2^{12} \quad \text{possibilities.}$$

For all corner cubies there are

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3^8 = 8\,! \times 3^8$$

possibilities.

Trying skillfully and introducing a complicated theory it can be figured out that only a twelfth of these patterns can be turned to the starting position.

So you get

$$\frac{8\,!\;\;12\,!}{2} \quad \frac{3^8}{3} \quad \frac{2^{12}}{2} = 43252\ oo327\ 44898\ 56ooo$$
$$\approx 4.3 \times 1o^{19}$$

- or in words : 43 sixtillion restorable patterns.

If a computer needs a millisecond to count one pattern it takes him 1.4 million years to do all of them.

These numbers are taken from David Singmaster's booklet "Notes on Rubik's Magic Cube". It also explains why only one twelfth of all patterns imaginable can be transformed into the starting position.

The chance to obtain the same position twice through shuffling randomly is almost zero. After shuffling you will get practically always a different pattern.

Listing the number of the different patterns still possible after a certain stage of the solution system, we obtain :

Cube shuffled	43 sixtillion	= $4.3\ 1o^{19}$
Upper layer arranged	still 16 billion	= $1.6\ 1o^{9}$
Middle layer arranged	still 62 2o8	≈ $6.2\ 1o^{4}$
Bottom face arranged	still 288	≈ $2.8\ 1o^{2}$
Bottom corners arranged	still 12	= $1.2\ 1o^{1}$

The dimension of the number of remaining patterns is cut into half with every step of the solution system.

After knowing a lot about the cube and being introduced to a way to rearrange the cube, there shall be hinted to other solution systems and other possibilities of puzzling with the cube.

3. Other solution systemy and cube puzzles

Besides the way decribed here there are many other possibilities to restore the cube.

Surely there are solution systems with which you can come to the rearranged cube in less moves.

For example you can put two stages together like rearranging corner and edge cubies in the bottom layer simultaneously. You can try and find a one-stage solution system for all patterns still possible.

Most of such sequences have less moves than the combination of sequences suggested in this booklet. But for all patterns you need different sequences. You have to remember or look up all these sequences which make the arrngement very complicated.

The solution system introduced in this booklet tries to balance the number of moves needed and simplicity.

After some practice it shall be possible to restore an arbitrarely shuffled cube in less than five minutes. As an average time you can expect two or three minutes, assuming the cube turns easily and the puzzler knows the solution system safely.

The fastest time known for this solution system is one and a half minutes.

Possibly some steps of the system can be improved. Eventually a completely different structure - for instance having completely different stages like : first all corners then all edges - is more efficient.

In controlling one solution system the possibilities of puzzling with the cube are not exhausted. For example you can take the problem : first restore the top, than the bottom and then the middle layer.

In a completely different type of problem one can try to bring the cube into another pattern.

With three moves you can create a two-colored pattern like a chess board on every face of the cube (see Fig. 73).

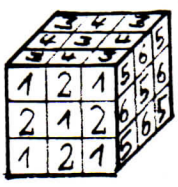


Fig. 73

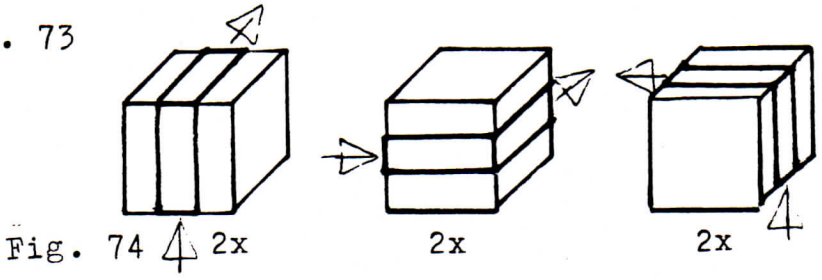


Fig. 74

The sequence to do this is simple (see Fig. 74). Each of the three middle layers is turned by 180°.

Another possibility of puzzling consist of looking for interesting - for instance symmetrical or simply colorful - patterns, eventually using only some of the layers.

Similar to our cube there are other cubes. Instead of a 3 x 3 x 3 cube you can investigate a 2 x 2 x 2 or 4 x 4 x 4 cube.

A 2 x 2 x 2 cube can be simulated with our cube in covering all facelets of the center and edge cubies with colored tape. Now you only have to shuffle the cube and restore it without watching the taped facelets. This problem can be solved with sequences from the previous chapters, but possibly there are "faster" solutions.

In order to puzzle with a 4 x 4 x 4 cube you better build one. Surely this takes a lot of skill in designing and fabricating it. You can also start puzzling it in your head.

Why shall one puzzle only with a cube ?

There are more regular polyhedra with layers turnable, for instance a tetrahedron like in Fig. 75.

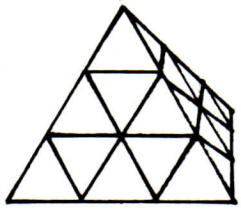


Fig. 75

There is a huge field of similar puzzles and similar problems with different demand to skill, creativity and endurance :

a puzzler's paradize.

We are still puzzling with cubes and polyhedra and we are grateful for hints and ideas :

Kersten Meier
4c Hulme, Escondido Village
Stanford, Ca. 943o5
U.S.A.

or

Henning-Storm-Str. 5
221 Itzehoe
West Germany

The 75-page booklet "Notes on Rubik's Magic Cube" can be ordered for 1.5o or $ 5.oo directly from :

David Singmaster
Mathematical Sciences and Computing
Polytechnic of the South Bank
London, SE1 OAA, England

- After reading this booklet you can rearrange the Rubik's Cube completely. The solution system is split into patrs to be understood easily. Many simple and difficult problems are provided to be solved by the reader and explained and solved afterwards.

- Throughout the world there have been sold four million Rubik's Cubes.

- In a competition in Hungary the winner needed only 4o seconds to rearrange his cube.

- In England for a cube fan a "Cubist's Thumb" was diagnozed - a complaint similar to the "Tennis Arm" of tennis players.